THE AREA MEASURE OF LOG-CONCAVE FUNCTIONS AND RELATED INEQUALITIES

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Abstract. On the class of log-concave functions on $\mathbb{R}^n$, endowed with a suitable algebraic structure, we study the first variation of the total mass functional, which corresponds to the volume of convex bodies when restricted to the subclass of characteristic functions. We prove some integral representation formulae for such first variation, which lead to define in a natural way the notion of area measure for a log-concave function. In the same framework, we obtain a functional counterpart of Minkowski first inequality for convex bodies; as corollaries, we derive a functional form of the isoperimetric inequality, and a family of logarithmic-type Sobolev inequalities with respect to log-concave probability measures. Finally, we propose a suitable functional version of the classical Minkowski problem for convex bodies, and prove some partial results towards its solution.

2010MSC: 26B25 (primary), 26D10, 52A20.

Keywords: Log-concave functions, convex bodies, area measure, isoperimetric inequality, log-Sobolev inequality, Minkowski problem.

Plan of the paper

1. Introduction

This article regards *log-concave* functions defined in $\mathbb{R}^n$, *i.e.* functions of the form

$$f : \mathbb{R}^n \to \mathbb{R}, \quad f = e^{-u},$$

where $u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex.

In the last decades the interest for log-concave functions has been considerably increasing, strongly motivated by the analogy between these objects and convex bodies (convex compact subsets of $\mathbb{R}^n$). The first breakthrough in the discovery of parallel behaviours of convex bodies and log-concave functions, was the *Prékopa-Leindler inequality*, named after the two Hungarian mathematicians who

proved it in the seventies [15, 17, 18, 19]. It states that, for any given functions $f, g, h \in L^1(\mathbb{R}^n; \mathbb{R}_+)$ which satisfy, for some $t \in (0,1)$, the pointwise inequality

$$h\big((1-t)x + ty\big) \geq f(x)^{1-t} g(y)^t \qquad \forall x, y \in \mathbb{R}^n \ ,$$

it holds

$$\int_{\mathbb{R}^n} h \geq \Big(\int_{\mathbb{R}^n} f\Big)^{1-t} \Big(\int_{\mathbb{R}^n} g\Big)^t \ . \tag{1.1}$$

Moreover, it was proved by Dubuc in [6] that the equality sign holds in (1.1) if and only if the functions f and g are log-concave and translates, meaning that $f(x) = g(x - x_0)$ for some $x_0 \in \mathbb{R}^n$. If K and L are measurable subsets of $\mathbb{R}^n$ such that also their Minkowski combination $(1-t)K + tL$ is measurable, by applying the Prékopa-Leindler inequality with f, g and h equal respectively to the characteristic functions of K, L and $(1-t)K + tL$, one obtains

$$V((1-t)K + tL) \geq V(K)^{1-t}\, V(L)^t \, .$$

This is an equivalent formulation of the classical *Brunn-Minkowski inequality*

$$V((1-t)K + tL)^{1/n} \geq (1-t)V(K)^{1/n} + tV(L)^{1/n} \ , \tag{1.2}$$

which holds with equality sign if and only if K and L belong to the class $\mathcal{K}^n$ of convex bodies in $\mathbb{R}^n$ and are homothetic, namely they agree up to a translation and a dilation.
The geometric inequality (1.2) is a cornerstone in Convex Geometry: it has many important consequences, such as the isoperimetric inequality for convex bodies, and the uniqueness issue in the solution of the Minkowski problem (see the survey paper [8] for an overview). On the other hand, in view of its functional form, inequality (1.1) is somehow more "flexible", and finds many applications in different fields, such as convex geometry, probability, mass transportation; we refer the reader to [2, 3, 24] for more information on Prékopa-Leindler inequality, including proofs and bibliographical references.
In the same way as (1.1) paraphrases (1.2) into the realm of functions, recently analytic versions of other geometric inequalities have been studied. In particular, we mention the so-called Blaschke-Santaló inequality, involving the product of the volume of a convex body and its polar: functional versions of it have been achieved by Ball [2], Artstein, Klartag and Milman [1], and Fradelizi and Meyer [7]. Let us also emphasize that a suitable notion of mean width for log-concave functions has been introduced by Klartag and Milman in [13], where some related Urysohn-type inequality are also proved; a short time ago, these topics have been further developed by Rotem in [21, 22].

In the same spirit, the aim of this paper is to cast some more light upon the geometry of log-concave functions, and to propose functional counterparts of some classical quantities and inequalities in Convex Geometry, that we briefly remind below (for more details, we refer to [23]).
Going back to the Brunn-Minkowsi inequality, let us recall that it admits a sort of "differential version", the so-called *Minkowski first inequality*, which reads

$$V_1(K, L) := \frac{1}{n} \lim_{t \to 0^+} \frac{V(K + tL) - V(L)}{t} \geq V(K)^{\frac{n-1}{n}} V(L)^{\frac{1}{n}} \qquad \forall K, L \in \mathcal{K}^n \ . \tag{1.3}$$

Inequality (1.3) can be easily obtained from (1.2), and it is in fact equivalent to it. Notice that, when L is the unit ball, $V_1(K, L)$ is just the perimeter of K, and (1.3) becomes the isoperimetric inequality in the class of convex bodies.
The term $V_1(K, L)$, which is one of the mixed volumes of K and L, admits a very simple and elegant integral representation:

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L \, d\sigma_K \ , \tag{1.4}$$

where h_L is the *support function* of L, and σ_K is the *area measure* of K. In view of (1.4), the measure σ_K is usually interpreted as the first variation of volume with respect to the Minkowski addition. The classical *Minkowski problem* consists in retrieving K from its surface area measure, and it is well-known that it admits a unique solution up to translations. More precisely, given any measure η on the unit sphere S^{n-1} which satisfies the compatibility conditions of having null barycenter and being not concentrated on an equator, there exists a convex body, unique up to translations, such that $\eta = \sigma_K$.

Our main goals are to provide a functional version of Minkowski first inequality (1.3), of the representation formula (1.4), and of the Minkowski problem. In this perspective, a crucial issue is to identify a good notion of "area measure" for a log-concave function. To that aim, we pursue a quite natural idea, namely we replace the volume of a convex body by the integral of a log-concave function: we set

$$J(f) = \int_{\mathbb{R}^n} f\, dx \ ,$$

and we compute the first variation of such integral functional with respect to suitable perturbations. Actually, log-concave functions can be equipped with two internal operations: a sum and a multiplication by positive reals, that will be denoted respectively by $\oplus$ and $\cdot$, and can be characterized as follows (see Section 2 for a more rigorous presentation). If $f = e^{-u}$ and $g = e^{-v}$ are log-concave functions and $\alpha, \beta > 0$, then

$$\alpha \cdot f \oplus \beta \cdot g := e^{-w} \,, \quad \text{where } w^* = \alpha u^* + \beta v^* \ . \tag{1.5}$$

Here * denotes as usual the Fenchel conjugate of convex functions. In other words, if we write a generic log-concave function as e^{-u}, the operations introduced in (1.5) are linear with respect to u^*. In particular, since the Fenchel conjugate of the indicatrix of a convex body is precisely its support function, one has

$$\alpha \cdot \chi_K \oplus \beta \cdot \chi_L = \chi_{\alpha K + \beta L} \ .$$

Therefore, definition (1.5) can be seen a natural extension to the class log-concave functions of the Minkowski structure on convex bodies.

In this framework, for a pair of log-concave functions f and g, we study the quantity

$$\delta J(f,g) := \lim_{t \to 0^+} \frac{J(f \oplus t \cdot g) - J(f)}{t} \ . \tag{1.6}$$

Let us point out that, red within this formalism, the above quoted works [13, 21, 22] are concerned precisely with the limit in (1.6), in the special case when f is equal to γ_n, the density of the Gaussian measure in $\mathbb{R}^n$. In fact, to some extent, γ_n plays the role of the unit ball in the class of log-concave functions. Thus, according to [13], the *mean width* of a log-concave function g is given by $\delta J(\gamma_n, g)$, by analogy with the mean width of a convex body K which is given by $V_1(B, K)$. We also mention the paper [12] by Klartag (see in particular §3), where a limit similar to (1.6) is considered, in the class of s–concave functions endowed with the appropriate algebraic operations, in order to derive several functional inequalities.

When f and g are arbitrary log-concave functions, the limit in (1.6) exists under the fairly weak condition $J(f) > 0$. In Section 3.1 we give a rigorous proof of this fact, already pointed out in [13], and we show that the condition $J(f) > 0$ is not necessary in the one dimensional case. Moreover we give simple examples which reveal that $\delta J(f,g)$ may become negative or $+\infty$ (indeed, whereas $V(K + tL)$ is a polynomial in t for every K and L in $\mathcal{K}^n$, this is no longer true in general for

$J(f \oplus t \cdot g))$. Then in Section 3.2 we compute $\delta J(f,g)$ in some special cases: the case when $f=g$, which brings into play the *entropy* of f:

$$\mathrm{Ent}(f) = \int_{\mathbb{R}^n} f \log f \, dx - J(f) \log J(f) \ ,$$

and the case when the logarithms of f and g are powers of support functions of convex bodies, which allows to recover an integral representation formula for the derivative of p-mixed volume due to Lutwak [16].

To go farther than these special cases, in Section 4 we come to the problem at the core of the paper, namely the problem of giving some general integral representation formula for $\delta J(f,g)$. We are able to achieve such a representation in two distinct settings: when the finiteness domains of $u=-\log f$ and $v=-\log g$ are the whole space $\mathbb{R}^n$, and when such domains are smooth strictly convex bodies. In both cases we have to assume further properties on u and v, concerning regularity, growth at the boundary of their domain, and strict convexity. To be more precise, our integral representation formulae are settled in the classes $\mathcal{A}'$, $\mathcal{A}''$ of log-concave functions $f=e^{-u}$ such that u belongs respectively to

$$\mathcal{L}' := \Big\{ u \in \mathcal{L} \ : \ \mathrm{dom}(u) = \mathbb{R}^n, \quad u \in \mathcal{C}^2_+(\mathbb{R}^n) \ , \quad \lim_{\|x\| \to +\infty} \tfrac{u(x)}{\|x\|} = +\infty \Big\} \, ,$$

$$\mathcal{L}'' := \Big\{ u \in \mathcal{L} \ : \ \mathrm{dom}(u) = K \in \mathcal{K}^n \cap \mathcal{C}^2_+, \quad u \in \mathcal{C}^2_+(\mathrm{int}(K)) \cap \mathcal{C}^0(K) \ , \quad \lim_{x \to \partial K} \|\nabla u(x)\| = +\infty \Big\} \ .$$

Here the notation $\mathcal{C}^2_+$, used for functions and sets, has the following standard meaning: when it is referred to a function u, it means that $u \in \mathcal{C}^2$ and the Hessian matrix of u is positive definite at each point; when it is referred to a convex body K, it means that $\partial K \in \mathcal{C}^2$ and the Gauss curvature is everywhere strictly positive.
After proving that $\mathcal{A}'$ and $\mathcal{A}''$ are both closed with respect to the operations $\oplus$ and $\cdot$ (see Lemma 4.9), we state our main results, which are valid under the assumption that the perturbation g is "controlled" by the perturbed function f (see Definition 4.4 for the precise statement of this assumption, which is not necessary in the one dimensional case). In Theorem 4.5 we prove that, when $f,g \in \mathcal{A}'$, $\delta J(f,g)$ is finite and is given by

$$\delta J(f,g) = \int_{\mathbb{R}^n} v^*(\nabla u(x)) f(x) \, dx \ . \tag{1.7}$$

In Theorem 4.6 we prove that, when $f,g \in \mathcal{A}''$, $\delta J(f,g)$ is finite and is given by

$$\delta J(f,g) = \int_K v^*(\nabla u(x)) f(x) \, dx + \int_{\partial K} h_L(\nu_K(x)) f(x) \, d\mathcal{H}^{n-1} \, , \tag{1.8}$$

where $K = \mathrm{dom}(u)$, ν_K is the unit outer normal to ∂K, $L = \mathrm{dom}(v)$, and h_L is the support function of L. The proof of these results is quite delicate and requires a careful analysis, see Section 4.
If we perform the change of variable $\nabla u(x) = y$ in (1.7), it becomes

$$\delta J(f,g) = \int_{\mathbb{R}^n} v^* \, d\mu(f) \ , \qquad d\mu(f) := f(y) e^{-\langle y, \nabla u^*(y) \rangle + u^*(y)} \ \det(\nabla^2 u^*(y)) \, dy \ . \tag{1.9}$$

Comparing (1.9) with (1.4), we are lead to identify the measure $\mu(f)$ as the *area measure* of a function f in the class A'. (Under this point of view, v^* plays the role of support function of g, as in [13]; this interpretation is quite natural if we remind that the algebraic structure we put on log-concave functions e^{-u} is linear with respect to u^*, in the same way as the Minkowski structure

on $\mathcal{K}^n$ is linear with respect to support functions). Similarly, with the changes of variable $\nabla u(x) = y$ and $\nabla \nu_K(y) = \xi$, (1.8) becomes

$$(1.10)\quad \delta J(f,g) = \int_{\mathbb{R}^n} v^*\, d\mu(f) + \int_{S^{n-1}} h_L\, d\sigma(f)\ , \quad d\mu(f) \text{ as above}, \quad d\sigma(f) := f(\nu_K^{-1}(\xi))\, d\sigma_K(\xi)\ .$$

Hence, within the class $\mathcal{A}''$, the notion of area measure of f is provided by the pair $(\mu(f), \sigma(f))$ (notice that the former is a measure on $\mathbb{R}^n$, the latter on S^{n-1}).

Having the above representation formulae at our disposal, we then turn attention to functional inequalities involving $\delta J(f,g)$. Our approach is similar to the one used by Klartag in [12] for the class of s-concave functions. In Section 5, we prove the following functional form of Minkowski first inequality (1.3) (see Theorem 5.1):

$$(1.11)\qquad \delta J(f,g) \geq J(f)\big[\log J(g) + n\big] + \mathrm{Ent}(f)\ ,$$

with equality sign if and only if there exists $x_0 \in \mathbb{R}^n$ such that $g(x) = f(x - x_0)$ $\forall x \in \mathbb{R}^n$. Loosely speaking, (1.3) can be proved taking the right derivative at $t = 0$ of both sides of the Brunn-Minkowski inequality (1.2), and inequality (1.11) is obtained by adapting this idea to the Prékopa-Leindler inequality, and using Dubuc's characterization of the equality case.
In Section 6 we show that, by combining the abstract inequality (1.11) with the above representation formulae for $\delta J(f,g)$, further functional inequalities come out.
Firstly, we define the *perimeter* of a function $f \in \mathcal{A}'$ in the natural way, that is as $P(f) := \delta J(f, \gamma_n)$, and we show that, under suitable assumptions, the following functional version of the isoperimetric inequality holds (see Proposition 6.2):

$$P(f) = \frac{1}{2}\int_{\mathbb{R}^n} \frac{\|\nabla f\|^2}{f}\, dx + (\log c_n)\, J(f) \geq nJ(f) + \mathrm{Ent}(f)\ ,$$

with equality sign if and only if there exists $x_0 \in \mathbb{R}^n$ such that $f(x) = \gamma_n(x - x_0)$ $\forall x \in \mathbb{R}^n$.
Then we derive a family of inequalities of logarithmic Sobolev type for probability measures ν with log-concave densities: under suitable assumptions on ν, a and h, we obtain (see Proposition 6.3)

$$(1.12)\qquad \int_{\mathbb{R}^n} a(h)\log a(h) d\nu - \Big(\int_{\mathbb{R}^n} a(h)\, d\nu\Big) \log\Big(\int_{\mathbb{R}^n} a(h)\, d\nu\Big) \leq \frac{1}{c}\int_{\mathbb{R}^n} \frac{(a'(h))^2}{a(h)} \|\nabla h\|^2\, d\nu\ .$$

In particular, by choosing $\nu = \gamma_n\, dx$ and $a(h) = h^2$, we recover Gross' logarithmic Sobolev inequality for the Gaussian measure. We point out that our approach allows much more general choices of ν and a; on the other hand, as a drawback, the validity of (1.12) is obtained under some further restrictions on h.

Finally, in Section 7 we move few steps towards the solution of the Minkowski problem for log-concave functions. As a natural extension of the Minkowski problem for convex bodies, such a problem can be formulated as follows: retrieve a log-concave function given its area measure. Clearly, in view of (1.9) and (1.10), the datum will consist of a single measure on $\mathbb{R}^n$ or of a pair of measures (the first on $\mathbb{R}^n$ and the second on S^{n-1}), depending on whether we want to solve the problem in the class $\mathcal{A}'$ or $\mathcal{A}''$, respectively. We establish a uniqueness result for both these problems (see Proposition 7.4), and we find some necessary conditions for the existence of a solution, which are quite similar to those afore mentioned about the classic Minkowski problem (see Proposition 7.2). However, differently from the case of convex bodies, it turns out that such conditions are in general *not* sufficient, as the analysis of the one dimensional case easily shows. Thus, at this stage, some substantial difference between the geometric and the functional setting emerges, which deserves in our opinion further investigation.

Acknowledgements. The authors wish to thank Shiri Artstein, Vitali Milman and Liram Rotem for their suggestions of useful bibliographical references related to the subject of this paper.

2. Preliminaries

2.1. **Notation and background.** We work in the n-dimensional Euclidean space $\mathbb{R}^n$, $n \geq 1$, endowed with the usual scalar product $\langle x, y\rangle$ and norm $\|x\|$; we set $B_r := \{x \in \mathbb{R}^n \,:\, \|x\| \leq r\}$.
For $m \leq n$, we indicate by $\mathcal{H}^m$ the m–dimensional Hausdorff measure; integration with respect to the Lebesgue measure $\mathcal{H}^n$ is abbreviated by dx.
We denote by $\mathcal{K}^n$ the class of convex bodies (compact convex sets) in $\mathbb{R}^n$, and by $\mathcal{K}^n_0$ the subclass of convex bodies K whose relative interior $\mathrm{int}(K)$ is nonempty. We indicate by $V(K) = \mathcal{H}^n(K)$ the n-dimensional volume of $K \in \mathcal{K}^n$.
Given $K \in \mathcal{K}^n_0$, we denote by ν_K its Gauss map, by $\sigma_K = (\nu_K)_\sharp(\mathcal{H}^{n-1} \llcorner \partial K)$ its surface area measure, and by $P(K) = \int_{S^{n-1}} d\sigma_K = \mathcal{H}^{n-1}(\partial K)$ its perimeter. We say that K is $\mathcal{C}^2_+$ if its boundary ∂K is of class $\mathcal{C}^2$ with strictly positive Gaussian curvature.
For any $K \in \mathcal{K}^n$, we adopt the standard notation h_K for the *support function* of K, defined by

$$h_K(x) := \sup_{y \in K} \langle x, y\rangle \qquad \forall x \in \mathbb{R}^n \ .$$

We recall that the *polar body* K^o of K is given by

$$K^o := \{y \in \mathbb{R}^n \ : \ \langle x, y\rangle \leq 1 \ \forall x \in K\} \ ;$$

if $0 \in \mathrm{int}(K)$, the support function of K agrees with the *gauge function* of K^o, namely

$$h_K(x) = \rho_{K^o}(x) := \inf\{t \geq 0 \ : \ x \in tK^o\} \ .$$

We denote by I_K and χ_K the *indicatrix function* and *characteristic function* of K, defined respectively by

$$I_K(x) := \begin{cases} 0 & \text{if } x \in K \\ +\infty & \text{if } x \notin K \,, \end{cases} \qquad \chi_K(x) := \begin{cases} 1 & \text{if } x \in K \\ 0 & \text{if } x \notin K \ . \end{cases}$$

Let $u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function. We set

$$\mathrm{dom}(u) = \{x \in \mathbb{R}^n \,:\, u(x) \in \mathbb{R}\} \,.$$

By the convexity of u, $\mathrm{dom}(u)$ is a convex set. We say that u is *proper* if $\mathrm{dom}(u) \neq \emptyset$. We say that u is *of class* $\mathcal{C}^2_+$ if it is twice differentiable on $\mathrm{int}(\mathrm{dom}(u))$, with a positive definite Hessian matrix. We denote by $\mathrm{epi}(u)$ the *epigraph* of u.
We recall that the *Fenchel conjugate* of u is the convex function defined by:

$$u^*(y) = \sup_{x \in \mathbb{R}^n} \langle x, y\rangle - u(x) \qquad \forall y \in \mathbb{R}^n .$$

On the class of convex functions from $\mathbb{R}^n$ to $\mathbb{R} \cup \{+\infty\}$, we consider the operation of *infimal convolution*, defined by

$$u \Box v(x) := \inf_{y \in \mathbb{R}^n} \big\{u(x-y) + v(y)\big\} \qquad \forall x \in \mathbb{R}^n \ , \tag{2.1}$$

and the following *right scalar multiplication* by a nonnegative real number α:

$$(u\alpha)(x) := \begin{cases} \alpha u\left(\frac{x}{\alpha}\right) & \text{if } \alpha > 0 \\ I_{\{0\}} & \text{if } \alpha = 0 \end{cases} \qquad \forall x \in \mathbb{R}^n \ . \tag{2.2}$$

Notice that these operations are convexity preserving, and that the function $I_{\{0\}}$ acts as the identity element in (2.1).

The proposition below gathers some elementary properties of the Fenchel conjugate, in particular about its behaviour with respect to the operations defined above. For the proof, we refer to [20].

Proposition 2.1. *Let $u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function. Then:*

(i) *it holds $u^*(0) = -\inf(u)$; in particular, $\inf(u) > -\infty$ implies u^* proper;*
(ii) *if u is proper, then $u^*(y) > -\infty$ $\forall y \in \mathbb{R}^n$;*
(iii) $\operatorname{dom}(u \Box v) = \operatorname{dom}(u) + \operatorname{dom}(v)$*;*
(iv) $(u \Box v)^* = u^* + v^*$*;*
(v) $(u\alpha)^* = \alpha u^*$*.*

Given a differentiable real valued function u on an open subset C of $\mathbb{R}^n$, the *Legendre conjugate* of the pair (C, u) is defined to be the pair (D, v), where D is the image of C through the gradient mapping ∇u, and

$$v(y) = \langle \nabla u^{-1}(y), y \rangle - u\big(\nabla u^{-1}(y)\big) \qquad \forall y \in D\ .$$

Such definition is well posed whenever, for any $y \in D$, the value of $\langle x, y\rangle - u(x)$ turns out to be independent from the choice of the point $x \in \nabla u^{-1}(y)$.
Following [20], we say that a pair (C, u) is a *convex function of Legendre type* if:

(a) C is a nonempty open convex set;
(b) u is differentiable and strictly convex on C;
(c) $\lim_i \|\nabla u(x_i)\| \to +\infty$ whenever $\{x_i\} \subset C$ is a sequence converging to some $x \in \partial C$.

Within the class of convex functions of Legendre type, Fenchel and Legendre conjugates may be identified according to Proposition below [20, Theorem 26.5].

Proposition 2.2. *Let $u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a closed convex function, and set $C := \operatorname{int}(\operatorname{dom}(u))$, $C^* := \operatorname{int}(\operatorname{dom}(u^*))$. Then (C, u) is a convex function of Legendre type if and only if (C^*, u^*) is. In this case, (C^*, u^*) is the Legendre conjugate of (C, u) (and conversely). Moreover, $\nabla u : C \to C^*$ is a continous bijection, with $(\nabla u)^{-1} = \nabla u^*$.*

2.2. **Functional setting.** Let us introduce the classes of functions we deal with throughout the paper.

Definition 2.3. We set:

$$\mathcal{L} := \left\{ u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \;\middle|\; u \text{ proper, convex, } \lim_{\|x\| \to +\infty} u(x) = +\infty \right\},$$

$$\mathcal{A} := \left\{ f : \mathbb{R}^n \to \mathbb{R} \;\middle|\; f = e^{-u}\,,\; u \in \mathcal{L} \right\}.$$

Below, we give some examples and basic properties of functions in $\mathcal{L}$; we show that, consequently, the class of log-concave functions $\mathcal{A}$ can be endowed with an algebraic structure which extends in a natural way the usual Minkowski structure on $\mathcal{K}^n$.

Example 2.4. (i) For any $K \in \mathcal{K}^n$, the function $u = I_K$ belongs to $\mathcal{L}$. Notice that $u^* = h_K$ belongs to $\mathcal{L}$ if and only if $0 \in \operatorname{int}(K)$, which shows that the class $\mathcal{L}$ is not closed under Fenchel transform.
(ii) For any $K \in \mathcal{K}^n$ with $0 \in \operatorname{int}(K)$, and any $p \in [1, +\infty)$, the function $u = \frac{1}{p} h_K^p$ belongs to $\mathcal{L}$. In particular, for any $p \in [1, +\infty)$, the function $u(x) = \frac{1}{p}\|x\|^p$ belongs to $\mathcal{L}$.

Lemma 2.5. *Let $u \in \mathcal{L}$. Then there exist constants a and b, with $a > 0$, such that*

$$u(x) \geq a\|x\| + b \qquad \forall\, x \in \mathbb{R}^n\,. \tag{2.3}$$

Moreover u^* is proper, and satisfies $u^*(y) > -\infty$ $\forall y \in \mathbb{R}^n$.

Proof. In order to show (2.3), assume first that $0 \in \mathrm{dom}(u)$. Let $r > 0$ be such that $u(x) \geq 1 + u(0)$ if $\|x\| \geq r$; for $\|x\| \geq r$, the convexity of u implies

$$u(x) \geq u(0) + \Big(u\big(\frac{rx}{\|x\|}\big) - u(0)\Big)\,\frac{\|x\|}{r} \geq u(0) + \frac{\|x\|}{r}\ .$$

Then, setting $m := \inf(u)$, it holds

$$u(x) \geq m - 1 + \frac{\|x\|}{r} \qquad \text{for } \|x\| \geq r\ .$$

Since the above inequality is verified for $\|x\| \leq r$ as well, it holds in $\mathbb{R}^n$. This shows that (2.3) is satisfied by taking $a = r^{-1}$ and $b = (m-1)$. In the general case, since u is proper, one can choose $x_0 \in \mathrm{dom}(u)$, and apply the above argument to the function $u(x - x_0)$, which yields

$$u(x) \geq a\|x + x_0\| + b \geq a\|x\| + b - a\|x_0\|\ .$$

The properties of u^* follow from Proposition 2.1 (i) and (ii). □

We now use Lemma 2.5 in order to prove that $\mathcal{L}$ is closed under the operations of infimal convolution and right scalar multiplication defined in (2.1) and (2.2).

Proposition 2.6. *Let $u, v \in \mathcal{L}$ and $\alpha, \beta \geq 0$. Then $(u\alpha)\Box(v\beta) \in \mathcal{L}$.*

Proof. From definition (2.2) it is immediate that $(u\alpha) \in \mathcal{L}$ for any $u \in \mathcal{L}$ and $\alpha \geq 0$. So we have just to show that $u\Box v$ belongs to $\mathcal{L}$ for any $u, v \in \mathcal{L}$. Set for brevity $w := u\Box v$. Clearly, w is a convex function defined in $\mathbb{R}^n$. Let us prove that w takes values into $\mathbb{R} \cup \{+\infty\}$, is proper and diverges as $\|x\| \to +\infty$.
By Proposition 2.1 (i) and (iv), we have

$$\inf w = -w^*(0) = -u^*(0) - v^*(0) = \inf(u) + \inf(v)\ .$$

Since $\inf(u), \inf(v) > -\infty$, we infer that $\inf(w) > -\infty$, which shows that w takes values into $\mathbb{R} \cup \{+\infty\}$.
By Proposition 2.1 (iii), $\mathrm{dom}(w) = \mathrm{dom}(u) + \mathrm{dom}(v)$, hence the properness of both u and v implies the same property for w.
Let $u(x) \geq a\|x\| + b$ and $v(x) \geq a'\|x\| + b'$ according to Lemma 2.5, and set $c := \min\{a, a'\} > 0$, $d := b + b'$. We have

$$\begin{aligned} w(x) &= \inf_{y \in \mathbb{R}^n} \big\{u(x - y) + v(y)\big\} \\ &\geq \inf_{y \in \mathbb{R}^n} \big\{a\|x - y\| + b + a'\|y\| + b'\big\} \\ &\geq \inf_{y \in \mathbb{R}^n} \big\{c(\|x - y\| + \|y\|) + d\big\} \geq c\|x\| + d\ . \end{aligned}$$

In particular, this implies that w diverges as $\|x\| \to +\infty$. □

We are now in a position to endow the class $\mathcal{A}$ with an addition and a multiplication by nonnegative scalars. These operations are internal to $\mathcal{A}$ thanks to Proposition 2.6.

Definition 2.7. Let $f = e^{-u}$, $g = e^{-v} \in \mathcal{A}$, and let $\alpha, \beta \geq 0$. We define

$$\alpha \cdot f \oplus \beta \cdot g = e^{-[(u\alpha)\Box(v\beta)]} , \tag{2.4}$$

which in explicit form reads

$$(\alpha \cdot f \oplus \beta \cdot g)(x) := \sup_{y\in\mathbb{R}^n} f\left(\frac{x-y}{\alpha}\right)^{\alpha} g\left(\frac{y}{\beta}\right)^{\beta} .$$

Remark 2.8. In view of the identities

$$u\Box v(x) = \inf\{\mu \ : \ (x,\mu) \in \mathrm{epi}(u) + \mathrm{epi}(v)\}$$
$$(u\alpha)(x) = \inf\{\mu \ : \ (x,\mu) \in \alpha\, \mathrm{epi}(u)\} ,$$

the functional operation in (2.4) has the following geometrical interpretation: it corresponds to the Minkowski combination with coefficients α and β of the epigraphs of u and v (as subsets of $\mathbb{R}^{n+1}$).

Next Proposition shows that, when restricted to suitable subclasses of $\mathcal{A}$, Definition 2.7 allows to recover different algebraic structures on convex bodies. Recall that (see [16]), for a fixed $p \in [1, +\infty)$, the p-sum of K and L with coefficients α and β is the convex body $\alpha \cdot_p K +_p \beta \cdot_p L$ defined by the equality

$$h^p_{\alpha\cdot_p K+_p\beta\cdot_p L} = \alpha h^p_K + \beta h^p_L .$$

Proposition 2.9. *Set*

$$\mathcal{L}_1 := \{h_{K^o} \ : \ K \in \mathcal{K}^n \ , \ 0 \in \mathrm{int}(K)\}$$
$$\mathcal{L}_q := \{\tfrac{1}{q}(h_{K^o})^q \ : \ K \in \mathcal{K}^n \ , \ 0 \in \mathrm{int}(K)\} , \quad q \in (1, +\infty) ,$$
$$\mathcal{L}_\infty := \{I_K \ : \ K \in \mathcal{K}^n\} .$$

The above subclasses of $\mathcal{L}$ are closed with respect to the operations defined in (2.1) *and* (2.2)*. More precisely, for any $\alpha, \beta \geq 0$, and any u, v belonging to the same class $\mathcal{L}_q$, it holds*

$$(u\alpha)\Box(v\beta) = \begin{cases} h_{K^o\cap L^o} & \textit{if } q=1,\ u=h_{K^\circ},\ v=h_{L^\circ}, \\ \frac{1}{p}(h_{(\alpha\cdot_p K+_p\beta\cdot_p L)^o})^p & \textit{with } p:=\frac{q}{q-1},\ \textit{if } q\in(1,+\infty),\ u=\frac{1}{q}(h_{K^\circ})^q,\ v=\frac{1}{q}(h_{L^\circ})^q, \\ I_{\alpha K+\beta L} & \textit{if } q=\infty,\ u=I_K,\ v=I_L.\ . \end{cases}$$

Proof. Let $u \in \mathcal{L}_q$. We have

$$u^* = \begin{cases} I_{K^o} & \text{if } q = 1 \\ \frac{1}{p}h^p_K & \text{if } q \in (1, +\infty) \\ h_K & \text{if } q = \infty . \end{cases} \tag{2.5}$$

In particular, in order to check the above expression of u^* in case $q \in (1, +\infty)$, one can apply with $\phi(s) = \frac{s^q}{q}$ the following identity holding for every increasing convex function ϕ (see *e.g.* [11]):

$$(\phi(h_{K^o}))^*(x) = \inf_{t\geq 0}\{\phi^*(t) + t h^*_{K^o}(\frac{x}{t})\} ;$$

this yields

$$\left(\frac{1}{q}(h_{K^o})^q\right)^*(x) = \inf_{\{t\geq 0\ :\ x\in tK^o\}} \{\frac{t^p}{p}\} = \frac{1}{p}\rho^p_{K^o}(x) = \frac{1}{p}h^p_K(x) .$$

Now, the statement of the Proposition follows easily from the computation of $\big((u\alpha)\Box(v\beta)\big)^*$. Indeed, by Proposition 2.1 (iv)-(v), it holds $\big((u\alpha)\Box(v\beta)\big)^* = \alpha u^* + \beta v^*$. According to (2.5), one has

$$\alpha u^* + \beta v^* = \begin{cases} \alpha I_{K^o} + \beta I_{L^o} = I_{K^o \cap L^o} = (h_{K^o\cap L^o})^* & \text{if } q = 1 \\ \frac{1}{p}\big[\alpha h_K^p + \beta h_K^p\big] = \frac{1}{p}\big[h_{\alpha \cdot_p K +_p \beta \cdot_p L}\big]^p = \big\{\frac{1}{q}\big[h_{(\alpha\cdot_p K +_p \beta \cdot_p L)^o}\big]^q\big\}^* & \text{if } q \in (1, +\infty) \\ \alpha h_K + \beta h_L = h_{\alpha K + \beta L} = (I_{\alpha K + \beta L})^* & \text{if } q = \infty\ . \end{cases}$$

□

3. Differentiability of the total mass functional

Definition 3.1. We call *total mass functional* the following integral

$$J(f) = \int_{\mathbb{R}^n} f(x)\, dx \qquad \forall f \in \mathcal{A}\ .$$

Remark 3.2. (i) The growth condition from below (2.3) satisfied by functions in $\mathcal{L}$ ensures that $J(f) \in [0, +\infty)$ for every $f \in \mathcal{A}$.
(ii) Clearly, when $f = \chi_K$, one has $J(f) = V(K)$.
(iii) If $f = e^{-u}$ is such that $J(f) = 0$, then $f = 0$ $\mathcal{H}^n$–a.e. in $\mathbb{R}^n$. This implies that the convex set $\mathrm{dom}(u)$ is Lebesgue negligible, and hence its dimension does not exceed $(n-1)$.

Remark 3.3. By the Prékopa–Leindler inequality, for every $f, g \in \mathcal{A}$ and for every $t \in [0, 1]$, it holds

$$J((1-t)\cdot f \,\oplus\, t \cdot g) \geq J(f)^{1-t}\, J(g)^t\ ,$$

with equality sign if and only if there exists $x_0 \in \mathbb{R}^n$ such that $g(x) = f\,(x - x_0)$ $\forall\, x \in \mathbb{R}^n$ (see [6, 8]). Consequently, for every fixed $f, g \in \mathcal{A}$, the functions $t \mapsto \log J(f \oplus t \cdot g)$ and $t \mapsto \log J\big((1-t)\cdot f \oplus t \cdot g\big)$ turn out to be concave respectively on $[0, +\infty)$ and on $[0, 1]$. We shall repeatedly exploit this concavity property in the sequel.

We are going to study the first variation of the total mass functional, with respect to the algebraic structure introduced in Definition 2.7.

Definition 3.4. Let $f, g \in \mathcal{A}$. Whenever the following limit exists

$$\lim_{t \to 0^+} \frac{J(f \oplus t \cdot g) - J(f)}{t}\ ,$$

we denote it by $\delta J(f, g)$, and we call it *the first variation of J at f along g.*

Remark 3.5. Let $f = \chi_K$ and $g = \chi_L$, with $K, L \in \mathcal{K}^n$. In this case $J(f \oplus t \cdot g) = V(K + tL)$ is a polynomial in t; its derivative at $t = 0^+$ is equal to n times the *mixed volume* $V_1(K, L)$, and admits the integral representation

$$\frac{d}{dt} V(K + tL)_{|_{t=0^+}} = n V_1(K, L) = \int_{S^{n-1}} h_L\, d\sigma_K\ . \tag{3.1}$$

Notice in particular that $\delta J(\chi_K, \chi_L)$ is nonnegative and finite, which is not always true in general for $\delta J(f, g)$ (*cf.* the examples given in Remark 3.8 below).

Subsection 3.1 below is devoted to prove that $\delta J(f, g)$ exists under the fairly weak hypothesis that $J(f)$ is strictly positive. Then in subsection 3.2 we show the explicit expression of $\delta J(f, g)$ in some relevant cases.

3.1. Existence of the first variation.

Theorem 3.6. *Let $f, g \in \mathcal{A}$, and assume that $J(f) > 0$. Then J is differentiable at f along g, and it holds*

$$\delta J(f, g) \in [-k, +\infty]\,, \tag{3.2}$$

being $k := [\inf(-\log g)]_+\, J(f)$. In dimension $n = 1$, the same conclusions continue to hold also when $J(f) = 0$.

Remark 3.7. We point out that the assumption $J(f) > 0$ it somehow technical; we believe that, when $J(f) = 0$, Theorem 3.6 is likely true not only in dimension $n = 1$ but also in higher dimensions (as it is suggested by the fact that the mixed volume $V_1(K, L)$ exists regardless the condition $V(K) > 0$).

Remark 3.8. Estimate (3.2) cannot be improved, as the following examples show.
(i) Let $f = e^{-u} \in \mathcal{A}$ with $J(f) > 0$, and $g = e^{-v}$, where $v(0) = 1$ and $v \equiv +\infty$ in $\mathbb{R}^n \setminus \{0\}$. Then $u\Box(vt)(x) = u(x) + t$, which implies

$$\delta J(f, g) = J(f) \cdot \lim_{t \to 0^+} \frac{e^{-t} - 1}{t} = -J(f) < 0\,.$$

(ii) Let $K, L \in \mathcal{K}^n$ with the origin in their interior, so that $u = h_K, v = h_L \in \mathcal{L}$, and take $f = e^{-u}, g = e^{-v}$. Then $u\Box(vt) = h_{K\cap L}$ (*cf.* Proposition 2.9), and therefore

$$\delta J(f, g) = \lim_{t \to 0^+} \left[\frac{1}{t} \int_{\mathbb{R}^n} \left(e^{-h_{K\cap L}} - e^{-h_L}\right) dx\right] = \begin{cases} 0 & \text{if } L \subseteq K \\ +\infty & \text{otherwise} \end{cases}.$$

Prior to the proof of Theorem 3.6, we state a preliminary lemma, which will be heavily exploited also in the next section.

Lemma 3.9. *Let $f = e^{-u}, g = e^{-v} \in \mathcal{A}$. For $t \geq 0$, set $u_t = u\Box(vt)$ and $f_t = e^{-u_t}$. Assume that $v(0) = 0$. Then, for every fixed $x \in \mathbb{R}^n$, $u_t(x)$ and $f_t(x)$ are respectively pointwise decreasing and increasing with respect to t; in particular it holds*

$$u_1(x) \leq u_t(x) \leq u(x) \quad \text{and} \quad f(x) \leq f_t(x) \leq f_1(x) \qquad \forall x \in \mathbb{R}^n\,,\ \forall t \in [0, 1]\,.$$

Proof. Given $t \geq 0$ and $\delta > 0$, let us show that $u_{t+\delta} \leq u_t$, *i.e.*

$$u\Box\big(v(t+\delta)\big) \leq u\Box(vt)\,.$$

If $t = 0$, the above inequality reduces to $u\Box(v\delta) \leq u$. This is readily checked: recalling definitions (2.1) and (2.2), from the assumption $v(0) = 0$ we deduce

$$u\Box(v\delta)(x) = \inf_{y \in \mathbb{R}^n} \left\{u(x-y) + \delta v\big(\tfrac{y}{\delta}\big)\right\} \leq u(x) \qquad \forall x \in \mathbb{R}^n\,.$$

If $t > 0$, for every $x \in \mathbb{R}^n$ we have

$$\begin{aligned} u\Box\big(v(t+\delta)\big)(x) &= \inf_{\xi \in \mathbb{R}^n} \Big\{u(x-\xi) + (t+\delta) v\big(\frac{\xi}{t+\delta}\big)\Big\} \\ &= \inf_{\xi \in \mathbb{R}^n} \Big\{u(x-\xi) + \inf_{y \in \mathbb{R}^n} \Big[t v\big(\frac{\xi - y}{t}\big) + \delta v\big(\frac{y}{\delta}\big)\Big]\Big\} \\ &= \inf_{y, z \in \mathbb{R}^n} \Big\{u(x-y-z) + t v\big(\frac{z}{t}\big) + \delta v\big(\frac{y}{\delta}\big)\Big\} \\ &= \big(u\Box(vt)\big)\Box(v\delta)(x) \leq u\Box(vt)(x). \end{aligned}$$

Thus u_t is monotone decreasing with respect to t, which immediately implies that $f_t = e^{-u_t}$ is monotone increasing. $\square$

Proof of Theorem 3.6. We set

$$u := -\log f\,, \qquad v := -\log g\,, \qquad f_t := f \oplus t \cdot g\,, \tag{3.3}$$

and

$$d := v(0)\,, \qquad \tilde v(x) := v(x) - d\,, \qquad \tilde g(x) := e^{-\tilde v(x)}\,, \qquad \tilde f_t := f \oplus t \cdot \tilde g\ . \tag{3.4}$$

Up to a translation of coordinates, we may also assume without loss of generality that $\inf(v) = v(0)$. Since by construction $\tilde v(0) = 0$, by Lemma 3.9 for every $x \in \mathbb{R}^n$ there exists $\tilde f(x) := \lim_{t\to 0^+} \tilde f_t(x)$ and it holds $\tilde f(x) \geq f(x)$. Moreover, by monotone convergence, we have $\lim_{t\to 0^+} J(\tilde f_t) = J(\tilde f)$. Since $f_t(x) = e^{-dt} \tilde f_t(x)$, we have

$$\frac{J(f_t) - J(f)}{t} = J(f)\,\frac{e^{-dt} - 1}{t} + e^{-dt}\,\frac{J(\tilde f_t) - J(f)}{t}\ . \tag{3.5}$$

Let us consider separately the two cases $J(\tilde f) > J(f)$ and $J(\tilde f) = J(f)$.
If $J(\tilde f) > J(f)$, then

$$\lim_{t\to 0^+} \frac{J(f_t) - J(f)}{t} = \lim_{t\to 0^+} \frac{J(\tilde f_t) - J(f)}{t} = +\infty\ ,$$

and the thesis of the theorem holds true.
If $J(\tilde f) = J(f)$, we further distinguish the following two subcases:

$$\exists t_0 > 0\ :\ J(\tilde f_{t_0}) = J(f) \qquad \text{or} \qquad J(\tilde f_t) > J(f)\ \forall t > 0\ .$$

In the former subcase, since by Lemma 3.9 $J(\tilde f_t)$ is a monotone increasing function of t, necessarily it holds $J(\tilde f_{t_0}) = J(f)$ for every $t \in [0, t_0]$. Hence the second addendum in the r.h.s. of (3.5) is infinitesimal, so that

$$\lim_{t\to 0^+} \frac{J(f_t) - J(f)}{t} = -dJ(f)$$

and the thesis of the theorem holds true.
In the latter subcase, we can write

$$\frac{J(\tilde f_t) - J(f)}{t} = \frac{\log(J(\tilde f_t)) - \log(J(f))}{t} \cdot \frac{J(\tilde f_t) - J(f)}{\log(J(\tilde f_t)) - \log(J(f))}\ . \tag{3.6}$$

Since $\log(J(\tilde f_t))$ is an increasing concave function of t (respectively by Lemma 3.9 and by the Prékopa–Leindler inequality, *cf.* Remark 3.2),

$$\exists \lim_{t\to 0^+} \frac{\log(J(\tilde f_t)) - \log(J(f))}{t} \in [0, +\infty]\,. \tag{3.7}$$

On the other hand,

$$\exists \lim_{t\to 0^+} \frac{J(\tilde f_t) - J(f)}{\log(J(\tilde f_t)) - \log(J(f))} = J(f) > 0. \tag{3.8}$$

From (3.6), (3.7), and (3.8), we infer that

$$\exists \lim_{t\to 0^+} \frac{J(\tilde f_t) - J(f)}{t} \in [0, +\infty]\,. \tag{3.9}$$

Combining (3.5) and (3.9), we deduce that

$$\exists \lim_{t\to 0^+} \frac{J(f_t) - J(f)}{t} \in [-\max\{d, 0\} J(f), +\infty]\,. \tag{3.10}$$

Finally, let us show that in the one–dimensional case $\delta J(f,g)$ exists also when $J(f)=0$. We keep definitions (3.3) and (3.4). Since by assumption $\mathrm{dom}(u)$ is a Lebesgue negligible convex set, it consists of exactly one point x_0. Then

$$u\Box(\tilde v t)(x) = u(x_0) + t\tilde v\big(\frac{x-x_0}{t}\big) \qquad \forall x\in\mathbb{R}\,,\ \forall t>0\,.$$

Hence

$$\lim_{t\to0^+}\frac{J(\tilde f_t)-J(f)}{t} = \lim_{t\to0^+} e^{-u(x_0)}\int_{\mathbb{R}} e^{-t\tilde v(x-x_0)}\,dx = e^{-u(x_0)}\mathcal{H}^1(\mathrm{dom}(v))\in[0,+\infty]\,, \tag{3.11}$$

where the last equality holds true by monotone convergence. Combining (3.5) and (3.11), we see that (3.10) remains true. $\square$

3.2. **Computation of the first variation in some special cases.** Firstly, we analyze the case $f=g$, and we show that $\delta J(f,f)$ admits a very simple representation in terms of the mass and the entropy of f, intended according to the definition below (*cf.* [14]).

Definition 3.10. For every $f\in\mathcal{A}$ with $J(f)>0$, we call *entropy* of f the following quantity:

$$\mathrm{Ent}(f) = \int_{\mathbb{R}^n} f\log f\,dx - J(f)\log J(f) \qquad \forall f\in\mathcal{A}\ .$$

Proposition 3.11. *For every $f\in\mathcal{A}$ with $J(f)>0$, it holds* $\mathrm{Ent}(f)\in(-\infty,+\infty)$ *and*

$$\delta J(f,f) = nJ(f) + \int_{\mathbb{R}^n} f\log f\,dx = \big(n+\log J(f)\big)J(f) + \mathrm{Ent}(f)\,. \tag{3.12}$$

Proof. Since $J(f)\in(0,+\infty)$ for every $f\in\mathcal{A}$, to prove the finiteness of $\mathrm{Ent}(f)$ we have just to show that

$$\int_{\mathbb{R}^n} f\log f\,dx \in(-\infty,+\infty)\ .$$

We set $u:=-\log f$ and $\Omega:=\{x\in\mathbb{R}^n\ :\ u(x)\le 0\}$ (which is possibly an empty set). It holds

$$\int_\Omega f\log f\,dx = -\int_\Omega fu\,dx < -\inf_\Omega(u)\int_\Omega f<+\infty\ ,$$

where in the last inequality we have used the boundedness of u from below on Ω and the finiteness of $J(f)$. On the other hand, we have

$$\int_{\mathbb{R}^n\setminus\Omega} f\log f\,dx = -\int_{\mathbb{R}^n\setminus\Omega} fu\,dx \ge -m\int_{\mathbb{R}^n\setminus\Omega} e^{-u(x)/2}\,dx > -\infty\ ,$$

where we have used the elementary inequality $te^{-t/2}\le m:=2/e$ holding for every $t\in\mathbb{R}_+$ and Lemma 2.5. So we have $J(f\log f)\in(-\infty,+\infty)$.

In order to prove the representation formula (3.12), assume first that $u\ge0$. Since $u\Box(ut)=u(1+t)$, we have

$$\begin{aligned}\frac{J(f\oplus t\cdot f)-J(f)}{t} &= \frac1t\left[(1+t)^n\int_{\mathbb{R}^n} e^{-(1+t)u}\,dx - \int_{\mathbb{R}^n} e^{-u}\,dx\right]\\ &= \left[\frac{(1+t)^n-1}{t}\right]\int_{\mathbb{R}^n} e^{-(1+t)u}\,dx + \int_{\mathbb{R}^n} e^{-u}\left(\frac{e^{-tu}-1}{t}\right)dx\,.\end{aligned}$$

Now (3.12) follows by passing to the limit as $t\to0^+$ (notice indeed that by the assumption $u\ge0$ one can apply the monotone convergence theorem).

In the general case when the assumption $u \geq 0$ is removed, we consider the function $\tilde{f} = e^{-\tilde{u}}$, where $\tilde{u} = u + c$ and $c = -\inf(u)$. One can easily check that $u\Box(ut) = -c(1+t) + \tilde{u}\Box(\tilde{u}t)$ and consequently $J(f \oplus t \cdot f) = e^{c(1+t)} J(\tilde{f} \oplus t \cdot \tilde{f})$. As $\tilde{u} \geq 0$, we know that $\delta J(\tilde{f}, \tilde{f})$ exists and it is finite, so the same is true for $\delta J(f, f)$. Moreover,

$$\delta J(f,f) = ce^c J(\tilde{f}) + e^c \delta J(\tilde{f}, \tilde{f}) = cJ(f) + e^c \left[nJ(\tilde{f}) - \int_{\mathbb{R}^n} e^{-(u+c)}(u+c)\,dx \right] = nJ(f) + \int_{\mathbb{R}^n} f \log f \,dx\,.$$

$\square$

Remark 3.12. By inspection of the above proof, one can readily check that also the left derivative

$$\lim_{t \to 0^-} \frac{J(f \oplus t \cdot f) - J(f)}{t}$$

exists and agrees with $\delta J(f, f)$.

Next we consider the case when f and g belong to the class $\mathcal{A}_q$ introduced in Proposition 2.9, and we show that $\delta J(f, g)$ can be written explicitly in integral form, by using the representation formula for p-mixed volumes given in [16].

Proposition 3.13. *Let $q \in (1, +\infty)$, and let $p := q/(q-1)$. Let $K, L \in \mathcal{K}^n$ with the origin in their interior, let $u := \frac{1}{q}(h_{K^o})^q$, $v := \frac{1}{q}(h_{L^o})^q$, and $f := e^{-u}$, $g := e^{-v}$. There exists a positive constant $c = c(n, q)$ such that*

$$J(f) = c(q,n)V(K) \tag{3.13}$$

$$\delta J(f,g) = \frac{c(q,n)}{n} \int_{S^{n-1}} h_L(\xi)^p (h_K(\xi))^{1-p}\, d\sigma_K(\xi)\ . \tag{3.14}$$

Proof. We set for brevity $a(t) = t^p/p$, so that $a^*(t) = t^q/q$. We have:

$$\begin{aligned}
J(f) = \int_{\mathbb{R}^n} e^{-a^*(h_{K^o})}\,dx \quad &= \int_0^1 \mathcal{H}^n\big(\{x \ :\ e^{-a^*(h_{K^o})(x)} > t\}\big)\,dt \\
&= \int_0^1 \mathcal{H}^n\big(\{x \ :\ h_{K^o}(x) < (a^*)^{-1}(-\log t)\}\big)\,dt \\
&= \int_0^1 \mathcal{H}^n\Big(\Big\{x \ :\ h_{K^o}\Big(\frac{x}{(a^*)^{-1}(-\log t)}\Big) < 1\Big\}\Big)\,dt \\
&= \int_0^1 \big((a^*)^{-1}(-\log t)\big)^n\,\mathcal{H}^n\big(\{y \ :\ h_{K^o}(y) < 1\}\big)\,dt \\
&= \Big\{ \int_0^1 \big((a^*)^{-1}(-\log t)\big)^n\,dt \Big\}\, V(K)\ ,
\end{aligned}$$

which proves (3.13) with $c(q,n) := \int_0^1 \big((a^*)^{-1}(-\log t)\big)^n\,dt$.
Now we recall from Proposition 2.9 that

$$f \oplus t \cdot g = e^{-\frac{1}{q}(h_{(K+_p t\cdot_p L)^o})^q}\ ,$$

which combined with (3.13) implies

$$\delta J(f,g) = c(q,n) \lim_{t\to 0^+} \frac{V(K +_p t \cdot_p L) - V(K)}{t}\ .$$

Then (3.14) follows from the representation formula for p-mixed volumes given in [16, (IIIp)]. $\square$

4. Integral representation of the first variation

In view of the examples in Section 3.2, it is natural to ask whether $\delta J(f,g)$ admits more in general some kind of integral representation. In this section we show that this is true when both f and g belong to suitable subclasses of $\mathcal{A}$.
Let us begin by introducing the measures which intervene in the representation formulae for $\delta J(f,g)$. Such measures can be viewed as the "first variation" of J in the class of log-concave functions, since they play for f the same role as the surface area measure for the volume in Convex Geometry. This fact emerges in a clear way by comparing the first variation of volume in (3.1) with Theorems 4.5 and 4.6 below.

Definition 4.1. Let $f = e^{-u} \in \mathcal{A}$. We set $\mu(f)$ the Borel measure on $\mathbb{R}^n$ defined by

$$\mu(f) := (\nabla u)_\sharp (f \mathcal{H}^n) \ .$$

When $\mathrm{dom}(u) = K \in \mathcal{K}^n$, we also set $\sigma(f)$ the Borel measure on S^{n-1} defined by

$$\sigma(f) := (\nu_K)_\sharp (f \mathcal{H}^{n-1} \llcorner \partial K) \ .$$

Next, we define the subclasses of $\mathcal{A}$ where our integral representation formualae are settled.

Definition 4.2. We set $\mathcal{A}', \mathcal{A}''$ the subclasses of $\mathcal{A}$ given by functions f such that $u = -\log f$ belongs respectively to

$$\mathcal{L}' := \Big\{ u \in \mathcal{L} \ : \ \mathrm{dom}(u) = \mathbb{R}^n, \quad u \in \mathcal{C}^2_+(\mathbb{R}^n) \ , \quad \lim_{\|x\| \to +\infty} \tfrac{u(x)}{\|x\|} = +\infty \Big\}$$

$$\mathcal{L}'' := \Big\{ u \in \mathcal{L} \ : \ \mathrm{dom}(u) = K \in \mathcal{K}^n \cap \mathcal{C}^2_+, \quad u \in \mathcal{C}^2_+(\mathrm{int}(K)) \cap \mathcal{C}^0(K) \ , \quad \lim_{x \to \partial K} \|\nabla u(x)\| = +\infty \Big\} \ .$$

Remark 4.3. Notice that, for any u belonging to $\mathcal{L}'$ or $\mathcal{L}''$, $(\mathrm{int}(\mathrm{dom}(u)), u)$ is a convex function of Legendre type, and u is *cofinite*, *i.e.* the domain of its Fenchel conjugate is the whole $\mathbb{R}^n$.

Finally, we introduce the concept of admissible perturbation.

Definition 4.4. We say that $g = e^{-v}$ is an *admissible perturbation* for $f = e^{-u}$ if

$$\exists\, c > 0 \ : \ \varphi - c\psi \text{ is convex} \ , \ \text{ where } \varphi = u^* \text{ and } \psi = v^* \ . \tag{4.1}$$

Our integral representation results read as follows.

Theorem 4.5. *Let $f, g \in \mathcal{A}'$, and assume that g is an admissible perturbation for f. Then $\delta J(f,g)$ is finite and is given by*

$$\delta J(f,g) = \int_{\mathbb{R}^n} \psi \, d\mu(f) \ , \tag{4.2}$$

where $\psi = v^*$.

Theorem 4.6. *Let $f, g \in \mathcal{A}''$, and assume that g is an admissible perturbation for f. Then $\delta J(f,g)$ is finite and is given by*

$$\delta J(f,g) = \int_{\mathbb{R}^n} \psi \, d\mu(f) + \int_{S^{n-1}} h_L \, d\sigma(f) \ , \tag{4.3}$$

where $\psi = v^*$ and $L = \mathrm{dom}(v)$.

Remark 4.7. For $n = 1$, (4.2) and (4.3) continue to hold, possibly as an equality $+\infty = +\infty$, if the assumption that g is an admissible perturbation for f is removed (see the Appendix for a proof).

Remark 4.8. Under the assumptions of Theorem 4.5 or Theorem 4.6, by using the definition of push-forward measure and the change of variables $\nabla u(x) = y$, one obtains

$$\int_{\mathbb{R}^n} \psi \, d\mu(f) = \int_{\mathrm{dom}(u)} \psi(\nabla u(x)) \, f(x) \, dx = \int_{\mathbb{R}^n} \psi(y) \, e^{-\langle y, \nabla \varphi(y) \rangle + \varphi(y)} \, \det(\nabla^2 \varphi(y)) \, dy \ .$$

Similarly, under the assumptions of Theorem 4.6, it holds

$$\int_{S^{n-1}} h_L \, d\sigma(f) = \int_{\partial K} h_L(\nu_K(x)) \, f(x) \, d\mathcal{H}^{n-1}(x) = \int_{S^{n-1}} h_L(\xi) \, f(\nu_K^{-1}(\xi)) \, \det(\nabla \nu_K^{-1}(\xi)) \, d\mathcal{H}^{n-1}(\xi) \ .$$

The proof of Theorems 4.5 and 4.6 is quite delicate and requires several preliminary lemmas, whose proof is postponed to the Appendix.
The first one establishes the closure of the two subclasses of $\mathcal{L}$ introduced in Definition 4.2 with respect to the operations of infimal convolution and right scalar multiplication.

Lemma 4.9. *Let u and v belong both to the same class $\mathcal{L}'$ or $\mathcal{L}''$ and, for any $t > 0$, set $u_t := u \square (vt)$. Then u_t belongs to the same class as u and v.*

We now turn attention to the behaviour of the functions $u_t = u \square (vt)$ with respect to the parameter t, more precisely regarding their pointwise convergence as $t \to 0^+$ (Lemma 4.10), and their differentiability in t (Lemma 4.11).

Lemma 4.10. *Let u and v belong both to the same class $\mathcal{L}'$ or $\mathcal{L}''$ and, for any $t > 0$, set $u_t := u \square (vt)$. Assume that $v(0) = 0$. Then*

(i) $\forall x \in \mathrm{dom}(u)$, $\quad \lim_{t \to 0^+} u_t(x) = u(x)$;
(ii) $\forall E \subset\subset \mathrm{dom}(u)$, $\quad \lim_{t \to 0^+} \nabla u_t(x) = \nabla u$ *uniformly on* E.

The following result is a key point in the proof of Theorems 4.5 and 4.6; it contains an explicit expression of the pointwise derivative of $u \square (ut)$ with respect to t.

Lemma 4.11. *Let u and v belong both to the same class $\mathcal{L}'$ or $\mathcal{L}''$ and, for any $t > 0$, let $u_t := u \square (vt)$. Then*

$$\forall x \in \mathrm{int}(\mathrm{dom}(u_t)) \, , \ \forall t > 0 \, , \qquad \frac{d}{dt} u_t(x) = -\psi\big(\nabla u_t(x)\big) \, , \ \text{where } \psi := v^* \ .$$

Next lemma provides a summability property of the Fenchel conjugate of $u = -\log f$ with respect to the measure $\mu(f)$ introduced in Definition 4.1.

Lemma 4.12. *Let $f = e^{-u} \in \mathcal{A}$, with $\varphi = u^* \geq 0$. Then $\varphi \in L^1(d\mu(f))$, namely*

$$\int_{\mathbb{R}^n} \varphi(\nabla u(x)) \, f(x) \, dx < +\infty \ .$$

Finally, when u, $v \in \mathcal{L}''$, we need an estimate for $u_t = u \square (vt)$ which will be exploited to deal with the boundary term in Theorem 4.6.

Lemma 4.13. *Let $u, v \in \mathcal{L}''$ and, for any $t > 0$, let $u_t = u \square (vt)$. Set $K := \mathrm{dom}(u)$, $L := \mathrm{dom}(v)$, $v_{\max} := \max_L v$, and $v_{\min} := \min_L v$. Then, for every $x \in K + tL$, there exists $y = y(x,t) \in K \cap (x - tL)$ such that*

$$tv_{\min} + u(y) \leq u_t(x) \leq tv_{\max} + u(y) \ .$$

Proof of Theorems 4.5 and 4.6.
We assume that either the hypotheses of Theorem 4.5 or the hypoteses of Theorem 4.6 are satisfied. Throughout the proof we set

$$f = e^{-u}\ , \qquad g = e^{-v}\ , \qquad \varphi = u^*, \qquad \psi = v^*\ , \qquad E = \mathrm{dom}(u)\ , \qquad F = \mathrm{dom}(v)\ ,$$

and, for every $t \geq 0$,

$$f_t = f \oplus t \cdot g\ , \qquad u_t = u \Box (vt)\ , \qquad \varphi_t = \varphi + t\psi\ , \qquad E_t = E + tF\ .$$

Let us point out that, under the assumptions of Theorem 4.5, we have $E = F = \mathbb{R}^n$, whereas, under the assumptions of Theorem 4.6, E and F are are convex bodies that will be named respectively K and L.
Further, we need to 'localize' our total mass functional: for every measurable set $A \subseteq \mathbb{R}^n$ and any function $h \in \mathcal{A}$, we set

$$J_A(h) := \int_A h\, dx\ .$$

For convenience, we divide the proof into several steps.

Step 1. Decomposition.

With the notation introduced above, we can write

$$J(f_t) - J(f) = J_E(f_t) - J_E(f) + J_{E_t \setminus E}(f_t).$$

We are going to prove the integral representation formulae (4.2) and (4.3) by showing that:
– under the assumptions of one among Theorems 4.5 and 4.6, it holds

$$\lim_{t \to 0^+} \frac{J_E(f_t) - J_E(f)}{t} = \int_{\mathbb{R}^n} \psi\, d\mu(f)\ ; \tag{4.4}$$

– under the assumptions of Theorem 4.6, it holds

$$\lim_{t \to 0^+} \frac{J_{E_t \setminus E}(f_t)}{t} = \int_{S^{n-1}} h_L\, d\sigma(f)\ . \tag{4.5}$$

Step 2. Reduction to the case $0 \in \mathrm{int}(F)$, $v(0) = 0$, $v \geq 0$, $\varphi \geq 0$, $\psi \geq 0$.

Assume that equalities (4.4) and (4.5) hold true (respectively under the assumptions of Theorems 4.5 or 4.6, and of Theorem 4.6), when all the conditions $0 \in \mathrm{int}(F)$, $v(0) = 0$, $v \geq 0$, $\varphi \geq 0$, $\psi \geq 0$ are satisfied.
In the general case, up to a translation of coordinates (which does not affect J), we may assume that $\inf v = v(0)$. Since by assumption v belongs to $\mathcal{L}'$ or $\mathcal{L}''$, its minimum is necessarily attained in the interior of its domain, so we have $0 \in \mathrm{int}(F)$. If $c := u(0)$ and $d := v(0)$, we set

$$\tilde u(x) := u(x) - c, \quad \tilde v(x) := v(x) - d, \quad \tilde\varphi(y) := (\tilde u)^*(y), \quad \tilde\psi(y) := (\tilde v)^*(y)$$

and

$$\tilde f = e^{-\tilde u}, \quad \tilde g = e^{-\tilde v}, \quad \tilde f_t := \tilde f \oplus t \cdot \tilde g\ .$$

By construction it holds $\mathrm{dom}(\tilde v) = F$, $\tilde v(0) = 0$, $\tilde v \geq 0$, $\tilde\varphi \geq 0$, $\tilde\psi \geq 0$. Then, taking also into account that $\mathrm{dom}(\tilde u) = E$, $\tilde\psi(y) = \psi(y) + d$, and $\tilde f = e^c f$, it holds

$$\lim_{t \to 0^+} \frac{J_E(\tilde f_t) - J_E(\tilde f)}{t} = \int_{\mathbb{R}^n} \tilde\psi\, d\mu(\tilde f) = e^c \int_{\mathbb{R}^n} \psi\, d\mu(f) + d e^c J_E(f) \tag{4.6}$$

and

$$\lim_{t \to 0^+} \frac{J_{E_t \setminus E}(\tilde f_t)}{t} = \int_{S^{n-1}} h_L\, d\sigma(\tilde f) = e^c \int_{S^{n-1}} h_L\, d\sigma(f)\ . \tag{4.7}$$

Now, since
$$f \oplus t \cdot g = e^{-(c+dt)} (\tilde{f} \oplus t \cdot \tilde{g}) \ ,$$
we may compute the left hand sides of (4.4) and (4.5) as derivatives of a product.
Using (4.6), we get
$$\lim_{t \to 0^+} \frac{J_E(f_t) - J_E(f)}{t} = -de^{-c} J_E(\tilde{f}) + e^{-c} \Big[e^c \int_{\mathbb{R}^n} \psi \, d\mu(f) + de^c J_E(f) \Big] = \int_{\mathbb{R}^n} \psi \, d\mu(f) \ .$$
Similarly, using (4.7), we get
$$\lim_{t \to 0^+} \frac{J_{E_t \setminus E}(f_t)}{t} = e^{-c} \cdot e^c \int_{S^{n-1}} h_L \, d\sigma(f) = \int_{S^{n-1}} h_L \, d\sigma(f) \ .$$

Step 3. For every $t > 0$, it holds
$$J_E(f_t) - J_E(f) = \int_0^t \Psi(s) \, ds \ , \tag{4.8}$$
where
$$\Psi(s) := \int_E \psi \, d\mu(f_s) \qquad \forall \, s \geq 0 \ . \tag{4.9}$$

Let $t > 0$ be fixed, and take $C \subset\subset E$. Thanks to the reduction $0 \in \mathrm{int}(F)$ made in Step 2, we have $C \subset\subset E_t$. Then by Lemma 4.11 it holds
$$\lim_{h \to 0} \frac{f_{t+h}(x) - f_t(x)}{h} = \psi(\nabla u_t(x)) \, f_t(x) \qquad \forall x \in C \ . \tag{4.10}$$
Moreover, thanks to the reduction $v(0) = 0$ made in Step 2, we can apply Lemma 3.9 and Lemma 4.10 (ii) to infer that, for every $s \in [0, 1]$, the nonnegative functions $\psi(\nabla u_s(x)) \, f_s(x)$ are bounded above on C by some continuous function independent of s. Then, by the pointwise convergence in (4.10), Lagrange theorem, and dominated convergence we infer
$$\lim_{h \to 0} \frac{J_C(f_{t+h}) - J_C(f_t)}{h} \, dx = \lim_{h \to 0} \int_C \frac{f_{t+h} - f_t}{h} \, dx = \int_C \psi(\nabla u_t) \, f_t \, dx \ .$$
So we have
$$J_C(f_t) - J_C(f) = \int_0^t \Big\{ \int_C \psi \, d\mu(f_s) \Big\} \, ds \ ,$$
which implies (4.8) by letting $C \uparrow E$.

Step 4. The function Ψ defined in (4.9) takes finite values at every $s \geq 0$.
Let $s > 0$. By the reduction $\varphi \geq 0$ made in Step 2, we have
$$s\Psi(s) \leq \int_{\mathbb{R}^n} (\varphi + s\psi) \, d\mu(f_s) = \int_{\mathbb{R}^n} u_s^*(\nabla u_s) f_s \, dx < +\infty \ ,$$
where the last inequality follows from Lemma 4.12 (which applies thanks to the conditions $\varphi, \psi \geq 0$).
Let now $s = 0$. Since by assumption g is an admissible perturbation for f, by (4.1) it holds
$$(\varphi - c\psi)(y) \geq (\varphi - c\psi)(0) + \langle y, \nabla \varphi(0) - c \nabla \psi(0) \rangle \ ,$$
so that
$$\psi(y) \leq c_1 + c_2 \varphi(y) + c_3 \|y\| \ ,$$
with
$$c_1 := \psi(0) - c^{-1} \varphi(0) \ , \qquad c_2 := c^{-1} \ , \qquad c_3 := c^{-1} \|\nabla \varphi(0) - c \nabla \psi(0)\| \ .$$

Therefore

$$\begin{aligned}\int_{\mathbb{R}^n} \psi(\nabla u(x))\, f(x)\, dx \quad &\leq c_1 \int_{\mathbb{R}^n} f(x)\, dx + c_2 \int_{\mathbb{R}^n} \varphi(\nabla u(x))\, f(x)\, dx + c_3 \int_{\mathbb{R}^n} \|\nabla u(x)\|\, f(x)\, dx \\ &=: c_1 I_1 + c_2 I_2 + c_3 I_3 \ .\end{aligned}$$

Let us show separately that each of the integrals I_j, $j = 1, 2, 3$, is finite. As already noticed in Remark 3.2 (i), the integral I_1 is finite for every $f \in \mathcal{A}$. The integral I_2 is finite by Lemma 4.12. Finally, in order to estimate the integral I_3, we use the coarea formula: if $m := \max_{\mathbb{R}^n} f$ it holds

$$I_3 = \int_{\mathbb{R}^n} \|\nabla f\|\, dx = \int_0^m \mathcal{H}^{n-1}\big(\partial\{f \geq s\}\big)\, ds \ . \tag{4.11}$$

According to Lemma 2.5, there exist constant a, b, with $a > 0$ such that

$$f(x) \leq g(x) := e^{-a|x\|-b} \ ,$$

which implies $\{f \geq s\} \subseteq \{g \geq s\}$, and in turn,

$$\mathcal{H}^{n-1}\big(\partial\{f \geq s\}\big) \leq \mathcal{H}^{n-1}\big(\partial\{g \geq s\}\big) = c(n)\Big(\frac{-\log s - b}{a}\Big)^{n-1} \ . \tag{4.12}$$

The finiteness of I_3 follows from (4.11) and (4.12).

Step 5. The function Ψ defined in (4.9) *is continuous at every $s > 0$, and it is continuous from the right at $s = 0$.*

Through the change of variable $\nabla u_s(x) = y$, we obtain

$$\Psi(s) = \int_E \psi(\nabla u_s(x))\, f_s(x)\, dx = \int_{\mathbb{R}^n} h(s, y)\, dy \ ,$$

with

$$h(s, y) := \psi(y) e^{\varphi_s(y) - \langle y, \nabla\varphi_s(y)\rangle} \det(\nabla^2\varphi_s)(y) \chi_{Q_s(y)} \ , \qquad Q_s := \nabla u_s(E) \ .$$

We now use the expansion

$$\det(\nabla^2\varphi_s) = \det(\nabla^2\varphi + s\nabla\psi) = \sum_{j=0}^n s^j D_j(\varphi, \psi) \ ,$$

where the mixed determinants $D_i(\varphi, \psi)$ are nonnegative functions of y independent of s. We infer that

$$\Psi(s) = \sum_{j=0}^n s^j \Psi_j(s) \ , \tag{4.13}$$

where

$$\Psi_j(s) := \int_{\mathbb{R}^n} h_j(s, y)\, dy \qquad h_j(s, y) := \psi(y) e^{\varphi_s - \langle y, \nabla\varphi_s\rangle} D_j(\varphi, \psi) \chi_{Q_s}(y) \ .$$

Let us prove the continuity of Ψ at a fixed $s_0 > 0$. In view of (4.13) it is enough to show that, for any fixed index $i \in \{0, 1, \ldots, n\}$, the function Ψ_i is continuous at s_0.

We begin by noticing that

$$\lim_{s \to s_0} \chi_{Q_s}(y) = \lim_{s \to s_0} \chi_{Q_{s_0}}(y) \qquad \forall y \in \mathbb{R}^n \ . \tag{4.14}$$

Indeed, when $E = F = \mathbb{R}^n$, (4.14) is trivially true since $Q_s = \mathbb{R}^n$ for every $s \geq 0$. Assume $E = K$ and $F = L$, with $K, L \in \mathcal{K}^n$. The reduction $0 \in \mathrm{int}(F)$ made in Step 2 ensures that $K \subset\subset K_{s_0}$, and hence by Lemma 4.10 (ii), we know that ∇u_s converge uniformly to ∇u_{s_0} on K. Therefore, the compact sets Q_s converge to Q_{s_0} in Hausdorff distance, which implies that the characteristic functions χ_{Q_s} converge to $\chi_{Q_{s_0}}$ in $L^1(\mathbb{R}^n)$, which in turn implies (4.14).

Using (4.14), we deduce that we have the pointwise convergence

$$\lim_{s \to s_0} h_i(s,y) = h_i(s_0,y) \qquad \forall y \in \mathbb{R}^n \ .$$

We claim that, as a consequence, $\Psi_i(s)$ tends to $\Psi_i(s_0)$ as $s \to s_0$ by dominated convergence. Indeed, let us show that $h_i(s,y)$ are bounded from above by a function in $L^1(\mathbb{R}^n)$ independent of s. By the reduction $v \geq 0$ made in Step 2, for any fixed $y \in \mathbb{R}^n$ the map

$$s \mapsto e^{\varphi_s(y) - \langle y, \nabla \varphi_s(y) \rangle} \tag{4.15}$$

is pointwise decreasing. Therefore, if we fix $\overline{s} \in (0, s_0)$, for any $s \geq \overline{s}$ it holds

$$\begin{aligned}
h_i(s,y) &\leq \psi e^{\varphi_{\overline{s}} - \langle y, \nabla \varphi_{\overline{s}} \rangle} D_i(\varphi, \psi) \\
&= \frac{1}{\overline{s}^i} \psi e^{\varphi_{\overline{s}} - \langle y, \nabla \varphi_{\overline{s}} \rangle} \overline{s}^i D_i(\varphi, \psi) \\
&\leq \frac{1}{\overline{s}^i} \psi e^{\varphi_{\overline{s}} - \langle y, \nabla \varphi_{\overline{s}} \rangle} \sum_{j=0}^{n} \overline{s}^j D_j(\varphi, \psi) \\
&= \frac{1}{\overline{s}^i} \psi e^{\varphi_{\overline{s}} - \langle y, \nabla \varphi_{\overline{s}} \rangle} \det(\nabla^2 \varphi_{\overline{s}}) \\
&\leq \frac{1}{\overline{s}^{i+1}} \varphi_{\overline{s}} e^{\varphi_{\overline{s}} - \langle y, \nabla \varphi_{\overline{s}} \rangle} \det(\nabla^2 \varphi_{\overline{s}}) \ ,
\end{aligned}$$

and the function in the last line belongs to $L^1(\mathbb{R}^n)$ by Lemma 4.12.
Let us now prove the continuity from the right of Ψ at $s = 0$. To that aim, in view of (4.13) is is enough to show that

$$\lim\nolimits_{s \to 0^+} \Psi_0(s) = \Psi(0) \tag{4.16}$$

$$\limsup\nolimits_{s \to 0^+} \Psi_i(s) < +\infty \qquad \forall\, i \in \{1, \dots, n\}. \tag{4.17}$$

To prove equality (4.16), we begin by noticing that, as $s \to 0^+$, the sets Q_s invade $\mathbb{R}^n$, meaning

$$\forall r > 0, \ \exists \overline{s} > 0 \ : \ Q_s \supseteq B_r \qquad \forall s \leq \overline{s} \ . \tag{4.18}$$

Indeed, when $E = F = \mathbb{R}^n$, (4.18) is trivially true since $Q_s = \mathbb{R}^n$ for every $s \geq 0$. Assume $E = K$ and $F = L$, with $K, L \in \mathcal{K}^n$, and let $r > 0$ be fixed. We have

$$Q_s = \nabla u_s(K) \supseteq \nabla u_s(C) \ , \qquad \text{with } C := \nabla u^{-1}(B_{2r}) \ . \tag{4.19}$$

Since $C \subset\subset K$ and $K \subset\subset K_s$ (the latter thanks to the reduction $0 \in \mathrm{int}(L)$ made in Step 2), by Lemma 4.10 (ii) we know that ∇u_s converge uniformly to ∇u on C. Therefore, the compact sets $\nabla u_s(C)$ converge to B_{2r} in Hausdorff distance, so that they contain B_r for s sufficiently small. Combined with (4.19), this implies (4.18).
Using (4.18), we deduce that we have the pointwise convergence

$$\lim_{s \to 0} h_0(s,y) = h_0(0,y) \qquad \forall y \in \mathbb{R}^n. \tag{4.20}$$

Now, by the monotonicity of the map (4.15), for any $s \geq 0$ it holds

$$h_0(s,y) \leq h_0(0,y) = \psi e^{\varphi - \langle y, \nabla \varphi \rangle} \det(\nabla^2 \varphi) \ , \tag{4.21}$$

and the last expression is in $L^1(\mathbb{R}^n)$ because we have proved in Step 4 that $\Psi(0)$ is finite.
In view of (4.20) and (4.21), (4.16) holds true by dominated convergence.
To prove (4.17) we notice that assumption (4.1) implies $\nabla^2 \psi \leq c^{-1} \nabla^2 \varphi$ and hence

$$D_i(\varphi, \psi) \leq D_i(\varphi, c^{-1} \varphi) \ .$$

This, combined with the monotonicity of the map (4.15), implies

$$h_i(s,y) \le \psi e^{\varphi - \langle y, \nabla\varphi\rangle} D_i(\varphi, c^{-1}\varphi) = \psi e^{\varphi - \langle y, \nabla\varphi\rangle} \gamma_i(c) \det(\nabla^2\varphi) \ ,$$

where the coefficients $\gamma_i(c)$ depend only on c. The last expression is in $L^1(\mathbb{R}^n)$ again by the finiteness of $\Psi(0)$, and (4.17) follows.

Step 6. Equality (4.4) *holds.*

The equality (4.8) proved in Step 3, together with the finiteness and continuity of $\Psi(s)$ for $s > 0$ proved respectively in Steps 4 and 5, gives

$$\Psi(s) = \frac{d}{dt} J_E(f_t)|_{t=s} \qquad \forall\, s > 0 \ . \tag{4.22}$$

Moreover, the continuity from the right of Ψ at $s = 0$ proved in Step 5 implies

$$\lim_{s\to 0^+} \Psi(s) = \Psi(0) = \int_{\mathbb{R}^n} \psi \, d\mu(f) \ . \tag{4.23}$$

Therefore,

$$\lim_{t\to 0^+} \frac{J_E(f_t) - J_E(f)}{t} = \frac{d}{dt} J_E(f_t)|_{t=0^+} = \lim_{s\to 0^+} \frac{d}{dt} J_E(f_t)|_{t=s} = \lim_{s\to 0^+} \Psi(s) = \int_{\mathbb{R}^n} \psi \, d\mu(f) \ . \tag{4.24}$$

Step 7. Under the assumptions of Theorem 4.6, equality (4.5) *holds.*

We define the map $m : S^{n-1} \times [0,t] \to K_t \setminus K$ by

$$m(\xi, s) := \nu_{K_s}^{-1}(\xi) = \nu_K^{-1}(\xi) + s\nu_L^{-1}(\xi) \ .$$

By the area formula [9, Section 3.1.5], we have

$$\int_{K_t \setminus K} f_t = \int_0^t \int_{S^{n-1}} f_t(m(\xi,s)) |\det Jm(\xi,s)| \, d\mathcal{H}^{n-1}(\xi) \, ds \ . \tag{4.25}$$

Let $(\xi, s) \in S^{n-1} \times [0,t]$ be fixed and let us compute $|\det Jm(\xi,s)|$. We choose an orthonormal basis $\{e_1, \dots, e_n\}$ of the tangent space $\xi^\perp \times \mathbb{R}$ to $S^{n-1} \times [0,t]$ given by

$$e_i = (v_i, 0) \quad i = 1, \dots, n-1 \ , \qquad e_n = (0, \dots, 0, 1) \ ,$$

where v_i are eigenvectors of the reverse Weingarten operator $\nabla \nu_{K_s}^{-1}(\xi)$. Then, denoting by $\rho_i(\xi, s)$ the corresponding eigenvalues (namely the principal radii of curvature of ∂K_s at ξ), it holds

$$\partial_{e_i} m(\xi,s) = \rho_i(\xi,s) e_i \quad i = 1, \dots, n-1 \ , \qquad \partial_{e_n} m(\xi,s) = \nu_L^{-1}(\xi) \ .$$

Hence

$$|\det Jm(\xi,s)| = \|\partial_{e_1} m(\xi,s) \wedge \cdots \wedge \partial_{e_n} m(\xi,s)\| = |\langle \xi, \nu_L^{-1}(\xi)\rangle| \cdot \prod_{i=1}^{n-1} \rho_i(\xi,s) = h_L(\xi) \det\big(\nabla \nu_{K_s}^{-1}(\xi)\big) \ ,$$

where the last equality holds because, by the reduction $0 \in \mathrm{int}(L)$ made in Step 2, we have $h_L \ge 0$. Now we recall that the reverse Weingarten operator of K_s is given by

$$\nabla \nu_{K_s}^{-1} = (h_{K_s})_{ij} + h_{K_s}\delta_{ij} \ ,$$

where indices i and j denote second order covariant derivation with respect to an orthonormal frame on S^{n-1}. Therefore, as $h_{K_s} = h_K + s h_L$, we have

$$\nabla \nu_{K_s}^{-1} = (h_K)_{ij} + h_K \delta_{ij} + s\big[(h_L)_{ij} + h_L \delta_{ij}\big] \ ,$$

and hence

$$|\det Jm(\xi,s)| = h_L(\xi)\Big[\det\big(\nabla\nu_K^{-1}(\xi)\big) + \sum_{i=1}^{n-1}\gamma_i(\xi)s^i\Big]\ , \tag{4.26}$$

where $\gamma_i(\xi)$ are continuous functions depending only on the curvatures of ∂K and ∂L at ξ.
Inserting (4.26) into (4.25) and dividing by t we obtain

$$\begin{aligned} \frac{1}{t}\int_{K_t\setminus K} f_t\,dx \ &= \frac{1}{t}\int_0^t\int_{S^{n-1}} f_t(m(\xi,s))h_L(\xi)\det\big(\nabla\nu_K^{-1}(\xi)\big)\,d\mathcal{H}^{n-1}(\xi)\,ds \\ &+\sum_{i=1}^{n-1}\frac{1}{t}\int_0^t s^i\Big\{\int_{S^{n-1}} f_t(m(\xi,s))h_L(\xi)\gamma_i(\xi)\,d\mathcal{H}^{n-1}(\xi)\Big\}\,ds\ . \end{aligned} \tag{4.27}$$

We observe that

$$\lim_{t\to 0^+}\sum_{i=1}^{n-1}\frac{1}{t}\int_0^t s^i\Big\{\int_{S^{n-1}} f_t(m(\xi,s))h_L(\xi)\gamma_i(\xi)\,d\mathcal{H}^{n-1}(\xi)\Big\} = 0\ . \tag{4.28}$$

Indeed, for every $i=1,\dots,n-1$, we have

$$\int_0^t s^i\Big\{\int_{S^{n-1}} f_t(m(\xi,s))h_L(\xi)\gamma_i(\xi)\,d\mathcal{H}^{n-1}(\xi)\Big\} \le (\sup_{\mathbb{R}^n} f_1)\int_{S^{n-1}} h_L\gamma_i\,d\mathcal{H}^{n-1}\int_0^t s^i\,ds\ ,$$

where we used the inequality $f_t(x)\le f_1(x)$ holding for every $x\in\mathbb{R}^n$ and every $t\in[0,1]$ by Lemma 3.9 (which applies thanks to the reduction $v(0)=0$ made in Step 2).
By (4.27) and (4.28), to conclude the proof of Step 7 it is enough to show that

$$\lim_{t\to 0^+}\frac{1}{t}\int_0^t\int_{S^{n-1}} f_t(m(\xi,s))h_L(\xi)\det\big(\nabla\nu_K^{-1}(\xi)\big)\,d\mathcal{H}^{n-1}(\xi)\,ds = \int_{S^{n-1}} h_L\,d\sigma(f)\ ,$$

or equivalently

$$\lim_{t\to 0^+}\frac{1}{t}\int_0^t\int_{S^{n-1}}\big[f_t(m(\xi,s)-f(m(\xi,0))\big]h_L(\xi)\det\big(\nabla\nu_K^{-1}(\xi)\big)\,d\mathcal{H}^{n-1}(\xi) = 0\ .$$

Such equality is clearly satisfied if

$$\lim_{t\to 0^+}\sup_{s\in[0,t],\ \xi\in S^{n-1}}\big|f_t(m(\xi,s))-f(m(\xi,0))\big| = 0\ . \tag{4.29}$$

Let $s\in[0,t]$ and $\xi\in S^{n-1}$. By Lemma 4.13 applied at the point $x:=m(\xi,s)\in\partial K_s\subset K_t$, there exists $y\in K\cap(x-tL)$ such that

$$tv_{\min}+u(y)\le u_t(m(\xi,s))\le tv_{\max}+u(y)\ .$$

Hence

$$tv_{\min}+u(y)-u(m(\xi,0))\le u_t(m(\xi,s))-u(m(\xi,0))\le tv_{\max}+u(y)-u(m(\xi,0))\ . \tag{4.30}$$

As $x\in m(\xi,0)+sL\subseteq m(\xi,0)+tL$, we have $m(\xi,0)\in K\cap(x-tL)$, and therefore

$$\|m(\xi,0)-y\|\le \mathrm{diam}\big(K\cap(x-tL)\big)\le t\mathrm{diam}(L)\ . \tag{4.31}$$

By (4.30), (4.31) and the uniform continuity of u on K, we infer that

$$\lim_{t\to 0^+}\sup_{s\in[0,t],\ \xi\in S^{n-1}}\big|u_t(m(\xi,s))-u(m(\xi,0))\big| = 0\ ,$$

and (4.29) follows.

Step 8: Conclusion.

Equalities (4.2) and (4.3) follow from Steps 1, 6, and 7. Moreover, the finiteness of $\Psi(0)$ proved in Step 4 implies that $\int_{\mathbb{R}^n} \psi \, d\mu(f) < +\infty$; on the other hand, for any $K, L \in \mathcal{K}^n$, one has $\int_{S^{n-1}} h_L \, d\sigma(f) < +\infty$. Therefore $\delta J(f,g)$ is finite.

5. The functional form of Minkowski first inequality

Minkowski first inequality states that

$$\lim_{t \to 0^+} \frac{V(K+tL) - V(K)}{t} = nV_1(K,L) \geq nV(K)^{1-\frac{1}{n}} V(L)^{\frac{1}{n}} \qquad \forall K, L \in \mathcal{K}_0^n \ , \tag{5.1}$$

with equality sign if and only if K and L are homothetic (see [23, Theorem 6.2.1]).
The main result of this section provides a functional version of such inequality:

Theorem 5.1. *Let $f, g \in \mathcal{A}$, and assume that $J(f) > 0$. Then*

$$\delta J(f,g) \geq J(f)\Big[\log J(g) + n\Big] + \mathrm{Ent}(f) \ , \tag{5.2}$$

with equality sign if and only if there exists $x_0 \in \mathbb{R}^n$ such that $g(x) = f(x - x_0)$ $\forall x \in \mathbb{R}^n$.

Remark 5.2. We point out that, by choosing $f = \gamma_n$, Theorem 5.1 allows to recover the Urysohn-type inequality for the mean width of a log-concave function proved in [13, Proposition 3.2] and [22, Theorem 1.4].

Before giving the proof of Theorem 5.1, let us present a straightforward consequence of it, which will be exploited in Section 7 in order to get uniqueness in the functional form of the Minkowski problem.

Corollary 5.3. *Let $f_1, f_2 \in \mathcal{A}$, with $J(f_1) = J(f_2) > 0$, and assume that*

$$\delta J(f_2, f_1) = \delta J(f_1, f_1) \qquad \text{and} \qquad \delta J(f_1, f_2) = \delta J(f_2, f_2) \ . \tag{5.3}$$

Then there exists $x_0 \in \mathbb{R}^n$ such that $f_2(x) = f_1(x - x_0)$ $\forall x \in \mathbb{R}^n$.

Proof. By the assumption $J(f_i) > 0$, we may apply inequality (5.2) (once with $f = f_1$ and $g = f_2$ and once with $f = f_2$ and $g = f_1$); since $J(f_1) = J(f_2)$, we get

$$\delta J(f_1, f_2) \geq nJ(f_1) + \int_{\mathbb{R}^n} f_1 \log f_1 \, dx \qquad \text{and} \qquad \delta J(f_2, f_1) \geq nJ(f_2) + \int_{\mathbb{R}^n} f_2 \log f_2 \, dx \ . \tag{5.4}$$

By assumption (5.3) and Proposition 3.11, the two inequalities in (5.4) may be rewritten respectively as

$$\delta J(f_2, f_2) \geq \delta J(f_1, f_1) \qquad \text{and} \qquad \delta J(f_1, f_1) \geq \delta J(f_2, f_2) \ ,$$

which implies that both hold with equality sign. Then f_1 and f_2 are translates of each other by Theorem 5.1. □

We now turn to the proof of Theorem 5.1. We need the following

Lemma 5.4. *Let $f, g \in \mathcal{A}$, and assume that $J(f) > 0$. Then*

$$\lim_{t \to 0^+} \frac{J\big((1-t) \cdot f \oplus t \cdot g\big) - J(f)}{t} = \delta J(f,g) - \delta J(f,f) \ .$$

Proof. For $t \in (0,1)$, we set

$$\alpha(t) := \frac{t}{1-t} \qquad \text{and} \qquad f_{\alpha(t)} := f \oplus \alpha(t) \cdot g \ .$$

Let us write

$$\frac{J\big((1-t)\cdot f \oplus t \cdot g\big) - J(f)}{t} = \frac{J\big((1-t)\cdot f_{\alpha(t)}\big) - J\big(f_{\alpha(t)}\big)}{t} + \frac{J\big(f_{\alpha(t)}\big) - J(f)}{t} \ , \tag{5.5}$$

and let us focus attention on the the first adddendum in the r.h.s. of (5.5).
For every fixed $t \in (0,1)$, we have

$$\frac{J\big((1-t)\cdot f_{\alpha(t)}\big) - J\big(f_{\alpha(t)}\big)}{t} = \frac{\gamma_t(t) - \gamma_t(0)}{t} \ ,$$

where the function γ_t is defined by

$$\gamma_t(s) := J\big((1-s)\cdot f_{\alpha(t)}\big) \qquad \forall s \in (0,1) \ .$$

In view of Proposition 3.11 and Remark 3.12, the function γ_t is differentiable on $(0,t)$, with

$$\begin{aligned}
\gamma_t'(s) &= -\delta J\big((1-s)\cdot f_{\alpha(t)}, (1-s)\cdot f_{\alpha(t)}\big) \\
&= -nJ\big((1-s)\cdot f_{\alpha(t)}\big) - \int_{\mathbb{R}^n} (1-s)\cdot f_{\alpha(t)} \log\big((1-s)\cdot f_{\alpha(t)}\big)\, dx \\
&= (1-s)^n \big[-nJ\big(f_{\alpha(t)}^{1-s}\big) - \int_{\mathbb{R}^n} f_{\alpha(t)}^{1-s} \log\big(f_{\alpha(t)}^{1-s}\big)\, dx \big] \ .
\end{aligned}$$

Then, for every fixed $t \in (0,1)$, we can apply Lagrange theorem to infer that there exists $\overline{s} \in (0,t)$ such that

$$\frac{J\big((1-t)\cdot f_{\alpha(t)}\big) - J\big(f_{\alpha(t)}\big)}{t} = \gamma_t'(\overline{s}) = (1-\overline{s})^n \big[-nJ\big(f_{\alpha(t)}^{1-\overline{s}}\big) - \int_{\mathbb{R}^n} f_{\alpha(t)}^{1-\overline{s}} \log\big(f_{\alpha(t)}^{1-\overline{s}}\big)\, dx \big] \ . \tag{5.6}$$

We are now ready to pass to the limit as $t \to 0^+$ in the r.h.s. of (5.5).
Concerning the first addendum, assume for a moment that the function $v := -\log g$ satisfies the condition $v(0) = 0$. In this case, by Lemma 3.9, as $t \to 0^+$ the functions $f_{\alpha(t)}(x)$ converge increasingly to some pointwise limit $\tilde f(x)$ (which is bounded above and below by some functions in $\mathcal{A}$). Then, by monotone convergence, taking also into account that $\overline{s} \to 0^+$ as $t \to 0^+$, we infer from (5.6) that

$$\lim_{t \to 0^+} \frac{J\big((1-t)\cdot f_{\alpha(t)}\big) - J\big(f_{\alpha(t)}\big)}{t} = -nJ\big(\tilde f\big) - \int_{\mathbb{R}^n} \tilde f \log \tilde f \, dx \in (-\infty, +\infty) \ . \tag{5.7}$$

Concerning the second addendum, differentiating a composition of functions shows immediately that

$$\lim_{t \to 0^+} \frac{J\big(f_{\alpha(t)}\big) - J(f)}{t} = \delta J(f,g) \ . \tag{5.8}$$

By combining (5.7) and (5.8), it is straightforward to conclude. Indeed, similarly as in the proof of Theorem 3.6, we may distinguish the two cases $J(\tilde f) > J(f)$ and $J(\tilde f) = J(f)$.
If $J(\tilde f) > J(f)$, the limit in (5.7) remains finite, whereas the limit in (5.8) becomes $+\infty$. Hence it holds

$$\lim_{t \to 0^+} \frac{J\big((1-t)\cdot f \oplus t \cdot g\big) - J(f)}{t} = \delta J(f,g) = +\infty \ ,$$

and the thesis of the lemma holds true.

If $J(\tilde{f}) = J(f)$, then $\tilde{f} = f$ $\mathcal{H}^n$-a.e., so that the r.h.s. of (5.7) agrees with $-\delta J(f,f)$, and the lemma follows summing up (5.7) and (5.8).
It remains to get rid of the assumption $v(0) = 0$. In the general case, we set as usual

$$d := v(0)\,, \qquad \tilde{v}(x) := v(x) - d\,, \qquad \tilde{g}(x) := e^{-\tilde{v}(x)}\ .$$

Since

$$(1-t)\cdot f \oplus t\cdot g = e^{-dt}\big((1-t)\cdot f \oplus t \cdot \tilde{g}\big)\ ,$$

we have

$$\frac{J\big((1-t)\cdot f\oplus t\cdot g\big) - J(f)}{t} = J(f)\,\frac{e^{-dt}-1}{t} + e^{-dt}\,\frac{J\big((1-t)\cdot f\oplus t\cdot \tilde{g}\big) - J(f)}{t}\,.$$

By passing to the limit as $t \to 0^+$, since $\tilde{v}(0) \geq 0$ by construction, we obtain

$$\lim_{t\to 0^+} \frac{J\big((1-t)\cdot f\oplus t\cdot g\big) - J(f)}{t} \geq -dJ(f) + \delta J(f,\tilde{g}) - \delta J(f,f)\ .$$

To conclude, it is enough to observe that $-dJ(f) + \delta J(f,\tilde{g}) = \delta J(f,g)$ (*cf.* (3.5)). □

Proof of Theorem 5.1. By the Prékopa-Leindler inequality, the function $\psi(t) := \log\big(J((1-t)\cdot f \oplus t\cdot g)\big)$ is concave on $[0,1]$ (*cf.* Remark 3.3). In particular, it holds

$$\psi(t) \geq \psi(0) + t\big[\psi(1) - \psi(0)\big] \qquad \forall t\in[0,1]\ . \tag{5.9}$$

As a consequence, the (right) derivative of the function ψ at $t = 0$ satisfies

$$\psi'(0) \geq \big[\psi(1) - \psi(0)\big]\ . \tag{5.10}$$

By Lemma 5.4, we have

$$\psi'(0) = \frac{\delta J(f,g) - \delta J(f,f)}{J(f)}\ .$$

Therefore (5.10) can be rewritten as

$$\frac{\delta J(f,g) - \delta J(f,f)}{J(f)} \geq \log\Big(\frac{J(g)}{J(f)}\Big)\ .$$

Inserting (3.12) into the above inequality, (5.2) is proved.
Finally, assume that $g(x) = f(x - x_0)$ for some $x_0 \in \mathbb{R}^n$. Then (5.2) holds with equality sign thanks to Proposition 3.11 and the invariance of J by translation of coordinates. Conversely, assume that (5.2) holds with equality sign. By inspection of the above proof one sees immediately that also inequality (5.10), and hence inequality (5.9), must hold with equality sign. This entails that the Prékopa-Leindler inequality holds as an equality, and therefore f and g agree up to a translation. □

6. Isoperimetric and log-Sobolev inequalities for log-concave functions

Let us now turn attention to some consequences of the results in Sections 4 and 5.
Motivated by the equality

$$\lim_{t\to 0^+}\frac{V(K+tB_1) - V(K)}{t} = P(K)\ ,$$

and having in mind that the Gaussian probability density

$$\gamma_n(x) := c_n e^{-\frac{\|x\|^2}{2}}\ , \qquad c_n := (2\pi)^{-\frac{n}{2}}\ ,$$

plays within the class $\mathcal{A}'$ the role of the unit ball in $\mathcal{K}^n$, we set the following

Definition 6.1. For any $f \in \mathcal{A}'$ with $J(f) > 0$, we define the *perimeter* of f as

$$P(f) := \delta J(f,\gamma_n)\ .$$

Similarly as Minkowski first inequality (5.1) (when applied with L equal to a ball B) implies the classical isoperimetric inequality

$$V(K)^{\frac{1}{n}} P(K)^{-\frac{1}{n-1}} \leq V(B)^{\frac{1}{n}} P(B)^{-\frac{1}{n-1}} \qquad \forall K \in \mathcal{K}^n_0 \ ,$$

Theorem 5.1 (when applied with $g = \gamma_n$ and combined with Theorem 4.5) yields the following functional version of the isoperimetric inequality:

Proposition 6.2. *Let $f = e^{-u} \in \mathcal{A}'$, and assume that $\varphi := u^*$ is uniformly strictly convex, namely*

$$\exists\, c > 0 \ : \ \nabla^2 \varphi(y) \geq c\,\mathrm{Id} \quad \forall y \in \mathbb{R}^n \ . \tag{6.1}$$

Then

$$P(f) = \frac{1}{2} \int_{\mathbb{R}^n} \frac{\|\nabla f\|^2}{f}\, dx + (\log c_n)\, J(f) \geq nJ(f) + \mathrm{Ent}(f) \ , \tag{6.2}$$

with equality sign if and only if there exists $x_0 \in \mathbb{R}^n$ such that $f(x) = \gamma_n(x - x_0)$ $\forall x \in \mathbb{R}^n$.

Proof. By the assumption (6.1), γ_n is an admissible perturbation for f according to Definition 4.4. Then, by Definition 6.1 and Theorem 4.5, one gets

$$P(f) = \delta J(f, \gamma_n) = \int_{\mathbb{R}^n} (\frac{1}{2}\|\nabla u\|^2 + \log c_n) f\, dx \ ,$$

which proves the first equality in (6.2). The subsequent inequality in (6.2) is obtained by applying Theorem 5.1 (simply take into account that $J(\gamma_n) = 1$). □

As a further application of our results, we now provide a generalized logarithmic Sobolev inequality for log-concave measures. After the pioneering result by Gross concerning the Gaussian measure [10], the validity of logarithmic Sobolev inequalities for more general probability measures, having in particular a log-concave density, has been investigated by several authors. We refer in particular to the paper [4] by Bobkov, where necessary and sufficient conditions are discussed.

Proposition 6.3. *Let $\nu = g\mathcal{H}^n = e^{-v}\mathcal{H}^n$ be a log-concave probability measure such that $g \in \mathcal{A}'$ and*

$$\nabla^2 v \geq c\,\mathrm{Id} \quad \text{ for some } c > 0 \ . \tag{6.3}$$

Let $a : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous increasing function with $a(0) = 0$.
Let h be a positive measurable function of class $\mathcal{C}^2(\mathbb{R}^n)$ which satisfies the conditions

$$\lim_{\|x\| \to +\infty} \frac{-\log(a(h)) + v}{\|x\|} = +\infty \quad \text{ and } \quad -c'\, \nabla^2 v \leq \nabla^2(\log(a(h)) < \nabla^2 v \quad \text{ for some } c' > 0 \ .$$

Then it holds

$$\int_{\mathbb{R}^n} a(h) \log a(h) d\nu - \Big(\int_{\mathbb{R}^n} a(h)\, d\nu \Big) \log \Big(\int_{\mathbb{R}^n} a(h)\, d\nu \Big) \leq \frac{1}{c} \int_{\mathbb{R}^n} \frac{(a'(h))^2}{a(h)} \|\nabla h\|^2\, d\nu \ . \tag{6.4}$$

Remark 6.4. The constant $\frac{1}{c}$ in the r.h.s. of (6.4) is non-optimal. Indeed, consider for instance the case when $g = \gamma_n$ (so that $c = 1$), and $a(h) = h^2$. Then (6.4) becomes

$$\int_{\mathbb{R}^n} h^2 \log(h^2) d\nu - \Big(\int_{\mathbb{R}^n} h^2\, d\nu \Big) \log \Big(\int_{\mathbb{R}^n} h^2\, d\nu \Big) \leq 4 \int_{\mathbb{R}^n} \|\nabla h\|^2\, d\nu \ , \tag{6.5}$$

and it is known that (6.5) holds true with 2 in place of 4 at the r.h.s. This assertion can be recovered by inspection of the proof below, since in this case the number t appearing in (6.9) equals $\frac{1}{2}$.

Remark 6.5. It is not surprising that, in order to have an inequality of logarithmic Sobolev type for the measure ν, condition (6.3) is needed; indeed, (6.3) can be related to the so-called Herbst necessary condition (see [4] for a more detailed discussion).

Proof of Proposition 6.3. Since $\int_{\mathbb{R}^n} g\,dx = 1$, inequality (5.2) reads

$$\delta J(f,g) \geq nJ(f) + \mathrm{Ent}(f)\ . \tag{6.6}$$

The computation of $J(f)$ and $\int_{\mathbb{R}^n} f\log f\,dx$ is straightforward:

$$J(f) = \int_{\mathbb{R}^n} a(h)\,d\nu\ , \qquad \int_{\mathbb{R}^n} f\log f\,dx = \int_{\mathbb{R}^n} \big(-v + \log a(h)\big)a(h)\,d\nu\ .$$

On the other hand, by the hypotheses made on h and g, the functions $f := a(h)g$ and g turn out to satisfy the assumptions of Theorem 4.5. Then, setting $\psi = v^*$, we have

$$\delta J(f,g) = \int_{\mathbb{R}^n} \psi\big(\nabla v - \nabla \log a(h)\big)a(h)\,d\nu\ .$$

Inserting the above expressions of $J(f)$, $\int_{\mathbb{R}^n} f\log f\,dx$, and $\delta J(f,g)$ into (6.6) leads to

$$\int_{\mathbb{R}^n} a(h)\log a(h) d\nu - \Big(\int_{\mathbb{R}^n} a(h)\,d\nu\Big)\log\Big(\int_{\mathbb{R}^n} a(h)\,d\nu\Big) \leq R(h)\ , \tag{6.7}$$

with

$$R(h) = \int_{\mathbb{R}^n} \Big[\psi\big(\nabla v - \nabla\log a(h)\big) + v - n\Big]a(h)\,d\nu\ .$$

Using the identity $v(x) = \langle x, \nabla v(x)\rangle - \psi(\nabla v(x))$, we may rewrite $R(h)$ as

$$R(h) = \int_{\mathbb{R}^n} \Big[\psi\big(\nabla v - \nabla\log a(h)\big) - \psi(\nabla v) + \langle x, \nabla v\rangle - n\Big]a(h)\,d\nu\ .$$

Now we observe that

$$\langle x, \nabla v\rangle a(h) g = -\langle x, \nabla g\rangle a(h) = -\mathrm{div}(xa(h)g) + \langle x, \nabla a(h)\rangle g + na(h)g\ ,$$

and

$$\int_{\mathbb{R}^n} \mathrm{div}(xa(h)g)\,dx = \lim_{r\to+\infty} r\int_{\partial B_r} a(h)g\,d\mathcal{H}^{n-1} = \lim_{r\to+\infty} r\int_{\partial B_r} f\,d\mathcal{H}^{n-1} = 0$$

(where the last equality is satisfied by the exponential decay of f at infinity, *cf.* Lemma 2.5). Therefore,

$$R(h) = \int_{\mathbb{R}^n} \Big[\psi\big(\nabla v - \nabla\log a(h)\big) - \psi(\nabla v) + \langle x, \nabla\log a(h)\rangle\Big]a(h)\,d\nu\ . \tag{6.8}$$

In view of (6.7) and (6.8), the statement is proved if the following pointwise inequality holds:

$$\psi\big(\nabla v - \nabla\log a(h)\big) - \psi(\nabla v) + \langle x, \nabla\log a(h)\rangle \leq \frac{1}{c}\|\nabla\log a(h)\|^2\ .$$

This is readily checked: indeed, setting $y := -\nabla\log a(h)$, by Lagrange theorem and assumption (6.3), there exist $t, s \in (0,1)$ such that

$$\begin{aligned} \psi\big(\nabla v + y\big) - \psi(\nabla v) - \langle x, y\rangle \ &= \langle \nabla\psi(\nabla v + ty), y\rangle - \langle\nabla\psi(\nabla v), y\rangle \\ &= \langle \nabla^2\psi(\nabla v + sty)ty, y\rangle \leq \frac{1}{c}\|y\|^2, \end{aligned} \tag{6.9}$$

and the proof is achieved. □

7. About the Minkowski problem

In this concluding section we move the first steps towards the solution of the functional Minkowski problem. In view of Theorems 4.5 and 4.6, its formulation within the class $\mathcal{A}'$ or $\mathcal{A}''$ reads as follows: find $f \in \mathcal{A}'$ such that

$$\mu(f) = m \ , \tag{7.1}$$

where m is a given positive Borel measure on $\mathbb{R}^n$, or find $f \in \mathcal{A}''$ such that

$$(\mu(f), \sigma(f)) = (m, \eta) \ , \tag{7.2}$$

where (m, η) are given positive Borel measures respectively on $\mathbb{R}^n$ and S^{n-1}. Here the measures $\mu(f)$ and $\sigma(f)$ are intended according to Definition 4.1.
We begin by the following simple observation.

Remark 7.1. We have the following finiteness necessary condition on the measures m and η, in order to solve the Minkowski problem with datum m or (m, η):

$$\int_{\mathbb{R}^n} dm < +\infty \,, \qquad \int_{S^{n-1}} d\eta < +\infty \,.$$

Indeed, if f belongs to $\mathcal{A}'$ or to $\mathcal{A}''$, we have

$$\int_{\mathbb{R}^n} d\mu(f) = J(f) < +\infty \,,$$

while, if $f \in \mathcal{A}''$ we have

$$\int_{S^{n-1}} d\sigma(f) \leq (\max_K f)\, \mathcal{H}^{n-1}(\partial K) < +\infty \,,$$

where $K = \mathrm{dom}(-\log f)$.

Next, we show that, for the solvability of (7.1), m must satisfy an equilibrium condition, which is completely analogous to the null barycenter property well-known in the classical Minkowski problem for convex bodies. The same holds true, for the solvability of (7.2), replacing m by the pair (m, η).

Proposition 7.2. *(i) For any $f \in \mathcal{A}'$, the measure $\mu(f)$ verifies*

$$\int_{\mathbb{R}^n} y \, d\mu(f)(y) = 0 \ .$$

(ii) For any $f \in \mathcal{A}''$, the measures $\mu(f)$ and $\sigma(f)$ verify

$$\int_{\mathbb{R}^n} y \, d\mu(f)(y) + \int_{S^{n-1}} y \, d\sigma(f)(y) = 0 \ .$$

Proof. Given a point $x_0 \in \mathbb{R}^n$ and a function $v \in \mathcal{L}$, we denote by $[v]_{x_0}$ the translated function $x \mapsto v(x - x_0)$. With this notation it is straightforward to check that, for any $u, v \in \mathcal{L}$, it holds

$$u \Box [v]_{x_o} = [u \Box v]_{x_0} \ . \tag{7.3}$$

Assume now that $f = e^{-u}$ belongs either to $\mathcal{A}'$ or to $\mathcal{A}''$. For any fixed $x_0 \in \mathbb{R}^n$ and any $\varepsilon > 0$, let us compute $\delta J(f, g_\varepsilon)$, where $g_\varepsilon = e^{-v_\varepsilon}$, being

$$v_\varepsilon(x) := \varepsilon u \Big(\frac{x - x_0}{\varepsilon} \Big) = [u\varepsilon]_{\frac{x_0}{\varepsilon}}(x) \qquad \forall \, x \in \mathbb{R}^n \ .$$

For any $t > 0$ one has

$$(v_\varepsilon t) = [u(t\varepsilon)]_{\frac{x_0}{\varepsilon}} \ ,$$

and hence, in view of (7.3),

$$u\Box(v_\varepsilon t) = [u\Box u(t\varepsilon)]_{\frac{x_0}{\varepsilon}} \ .$$

Therefore,

$$\delta J(f, g_\varepsilon) = \lim_{t\to 0^+} \frac{J(e^{-u\Box u(t\varepsilon)}) - J(f)}{t} = \varepsilon \lim_{t\to 0^+} \frac{J(e^{-u\Box u(t\varepsilon)}) - J(f)}{t\varepsilon} = \varepsilon \delta J(f, f) \ . \tag{7.4}$$

On the other hand, we observe that

$$v_\varepsilon^*(y) = \langle x_0, y\rangle + \varepsilon u^*(y) \qquad \text{and} \qquad \mathrm{dom}(v_\varepsilon) = x_0 + \varepsilon \mathrm{dom}(u) \ .$$

Therefore, if $f \in \mathcal{A}'$, by applying Theorem 4.5 we get

$$\delta J(f, g_\varepsilon) = \int_{\mathbb{R}^n} \langle x_0, y\rangle \, d\mu(f)(y) + \varepsilon \int_{\mathbb{R}^n} u^*(y) \, d\mu(f)(y) \ ; \tag{7.5}$$

similarly, if $f \in \mathcal{A}''$, by applying Theorem 4.6 we get

$$\begin{aligned} \delta J(f, g_\varepsilon) &= \int_{\mathbb{R}^n} \langle x_0, y\rangle \, d\mu(f)(y) + \varepsilon \int_{\mathbb{R}^n} u^*(y) \, d\mu(f)(y) \\ &\quad + \int_{S^{n-1}} \langle x_0, y\rangle \, d\sigma(f)(y) + \varepsilon \int_{S^{n-1}} h_{\mathrm{dom}(u)}(y) \, d\sigma(f)(y) \ . \end{aligned} \tag{7.6}$$

We now observe that the following terms, which appear multiplied by ε in (7.4), (7.5) and (7.6), are finite:

$$\delta J(f, f) \ , \quad \int_{\mathbb{R}^n} u^*(y) \, d\mu(f)(y) \ , \quad \int_{S^{n-1}} h_{\mathrm{dom}(u)}(y) \, d\sigma(f)(y)$$

(recall in particular Proposition 3.11 and Lemma 4.12). Then the statement follows by combining (7.4) with (7.5) or (7.6), in the limit as $\varepsilon \to 0^+$. □

Remark 7.3. We observe that the conditions expressed by Remark 7.1 and Lemma 7.2 are in general not sufficient for the solvability of the Minkowski problem within one of the classes $\mathcal{A}'$ or $\mathcal{A}''$. Indeed, assume for instance that $n = 1$ and consider the Minkowski problem in $\mathcal{A}'$: given an absolutely continuous measure on $\mathbb{R}$ with a positive continuous density m, satisfying the necessary conditions $\int_{\mathbb{R}} m(y)\, dy < +\infty$ and $\int_{\mathbb{R}} ym(y)\, dy = 0$, it amounts to finding a function $\varphi \in \mathcal{C}^2_+(\mathbb{R})$, with $u = \varphi^* \in \mathcal{L}'$, solving the second order o.d.e.

$$e^{\varphi(y) - y\varphi'(y)} \varphi''(y) = m(y) \qquad \forall y \in \mathbb{R} \ . \tag{7.7}$$

We observe that, if φ is a solution to (7.7), for any $\alpha \in \mathbb{R}$, also $\varphi + \alpha y$ is a solution. Therefore, we may assume with no loss of generality that $\varphi'(0) = 0$, and write the unique solution to (7.7) with initial datum at $y = 0$ as

$$\varphi(y) = \varphi(0) - y \int_0^y \frac{\log(e^{\varphi(0)} - M(t)) - \varphi(0)}{t^2} \, dt \ , \qquad \text{where } M(t) := \int_0^t sm(s) \, ds \ . \tag{7.8}$$

Now, in order that $u = \varphi^* \in \mathcal{L}'$, we have to impose that $\frac{\varphi(y)}{y}$ diverges as $|y| \to +\infty$. Such condition can be satisfied (by inspection of (7.8)) only if

$$e^{\varphi(0)} = M_\infty := \int_0^{+\infty} sm(s) \, ds \ . \tag{7.9}$$

By (7.8) and (7.9), it holds

$$\lim_{y\to+\infty} \frac{\varphi(y)}{y} = \lim_{y\to+\infty} \int_0^y \frac{\log(M_\infty - M(t)) - \log M_\infty}{t^2} \, dt \ .$$

It is quite easy to construct explicit examples of positive continuous functions m, with finite integral and zero barycenter, such that limit at the r.h.s. of the above equality remains finite. For such a datum m, the Minkowski problem does not admit solutions in $\mathcal{A}'$.

In view of the above Remark, and since in higher dimensions equality (7.1) does not correspond any longer to an o.d.e., but rather to a Monge-Ampère type equation, proving a general existence result for the functional Minkowski problem seems to be a quite delicate task. On the other hand, as a consequence of Corollary 5.3, we are able to prove that uniqueness (up to translations) holds true, in both the cases of $\mathcal{A}'$ and $\mathcal{A}''$.

Proposition 7.4. *Let $f_1, f_2 \in \mathcal{A}$ satisfy one of the following conditions:*

$$f_i \in \mathcal{A}' \quad i = 1, 2, \qquad \text{and} \qquad \mu(f_1) = \mu(f_2) \tag{7.10}$$

or

$$f_i \in \mathcal{A}'' \quad i = 1, 2, \qquad \text{and} \qquad \mu(f_1) = \mu(f_2),\ \sigma(f_1) = \sigma(f_2)\ . \tag{7.11}$$

Then there exists $x_0 \in \mathbb{R}^n$ such that $f_2(x) = f_1(x - x_0)$.

Proof. Firstly notice that the equality $\mu(f_1) = \mu(f_2)$ implies $J(f_1) = J(f_2)$. Moreover the assumption $f_i \in \mathcal{A}'$ (or $f_i \in \mathcal{A}''$) implies that $J(f_i) > 0$. If (7.10) holds, by Theorem 4.5 one has

$$\delta J(f_1, g) = \delta J(f_2, g) \qquad \forall g \in \mathcal{A}''\ .$$

In particular, taking $g = f_1$ or $g = f_2$, one sees that condition (5.3) is satisfied. Therefore, we are in a position to apply Corollary 5.3, and the statement follows. If (7.11) holds, the proof is exactly the same by using Theorem 4.6 in place of Theorem 4.5. $\square$

8. Appendix

This appendix contains the proofs of some results stated in Section 4, precisely all the preliminary lemmas used in the proof of Theorems 4.5 and 4.6, and the claim made in Remark 4.7.

Proof of Lemma 4.9. It is immediate to check that the classes $\mathcal{L}'$ and $\mathcal{L}''$ are closed by right multiplication by a positive scalar. Let us show that each of them is closed also by infimal convolution.
(i) Let $u, v \in \mathcal{L}'$, set $\varphi := u^*$, $\psi := v^*$, and $w := u \square v$.
By Proposition 2.1 (iii), it holds $\mathrm{dom}(w) = \mathrm{dom}(u) + \mathrm{dom}(v) = \mathbb{R}^n$.
The condition of having a superlinear growth at infinity is equivalent to the condition of being cofinite [5, Proposition 3.5.4], and the latter is clearly closed by infimal convolution in view of the equality $w^* = \varphi + \psi$ holding by Proposition 2.1 (iv). Therefore, w has superlinear growth at infinity. Since $(\mathbb{R}^n, u)$ and $(\mathbb{R}^n, v)$ are convex functions of Legendre type, with $u, v \in \mathcal{C}^2_+$, the mappings ∇u and ∇v are $\mathcal{C}^1$ bijections from $\mathbb{R}^n$ to $\mathbb{R}^n$, with a nonsingular Jacobian. Therefore also their inverse maps, which by Proposition 2.2 are precisely $\nabla\varphi$ and $\nabla\psi$, are $\mathcal{C}^1$ bijections from $\mathbb{R}^n$ to $\mathbb{R}^n$, and the same holds true for their sum. Hence $(\mathbb{R}^n, \varphi + \psi)$ is a convex function of Legendre type, with $\varphi + \psi$ of class $\mathcal{C}^2_+$. In turn, this implies that the Legendre conjugate of $(\mathbb{R}^n, \varphi + \psi)$, namely $(\mathbb{R}^n, w)$, is a convex function of Legendre type, with w of class $\mathcal{C}^2_+$.
(ii) Let $u, v \in \mathcal{L}''$, and set $K := \mathrm{dom}(u)$, $L := \mathrm{dom}(v)$, φ, ψ, and w as above.
By Proposition 2.1 (iii), it holds $\mathrm{dom}(w) = K + L \in \mathcal{K}^n \cap \mathcal{C}^2_+$.
Since u and v are of class $\mathcal{C}^2_+$, and their gradients diverge at the boundary of their domains, $(\mathrm{int}(K), u)$ and $(\mathrm{int}(L), v)$ are convex functions of Legendre type, and the mappings ∇u and ∇v are

$\mathcal{C}^1$ bijections respectively from K and L onto $\mathbb{R}^n$. Hence, similarly as above, we may apply Proposition 2.2 to infer that $(\mathbb{R}^n, \varphi)$, $(\mathbb{R}^n, \psi)$, and hence $(\mathbb{R}^n, \varphi + \psi)$, are convex functions of Legendre type, with $\varphi + \psi$ of class $\mathcal{C}^2_+$. This yields that $(\mathbb{R}^n, w)$ is a convex function of Legendre type, with w of class $\mathcal{C}^2_+$.

It remains to check that w is continuous up to $\partial(K + L)$. To this end we are going to use as a crucial tool the identity

$$(8.1)\quad u \Box v(x) = \inf_{x_1 + x_2 = x} \{u(x_1) + v(x_2)\} = u\big(\nu_K^{-1}(\nu_{K+L}(x)\big) + v\big(\nu_K^{-1}(\nu_{K+L}(x)\big) \quad \forall x \in \partial(K + L) \ ,$$

which follows from the definition of infimal convolution and the assumption $\partial K, \partial L \in \mathcal{C}^2_+$.

Let $\overline{x} \in \partial(K + L)$, and let us show that for every sequence of points $x^h \in K + L$ such that $x^h \to \overline{x}$, it holds

$$(8.2)\qquad \lim_h u \Box v(x^h) = u \Box v(\overline{x}) \ .$$

Up to passing to a (not relabeled) subsequence, we may assume that one of the following two cases occurs:

$$x^h \in \partial(K + L) \ \forall h \qquad \text{or} \qquad x^h \in \mathrm{int}(K + L) \ \forall h \ .$$

Consider first the case $x^h \in \partial(K + L) \ \forall h$. Let us write the identity (8.1) at x^h

$$u \Box v(x^h) = u\big(\nu_K^{-1}(\nu_{K+L}(x^h)\big) + v\big(\nu_K^{-1}(\nu_{K+L}(x^h)\big) \quad \forall h \ ,$$

and then let us pass to the limit in h. Since by hypothesis the Gauss maps ν_K, ν_L and their inverse are continuous, and u, v are continuous up to ∂K, ∂L, we get

$$\lim_h u \Box v(x^h) = u\big(\nu_K^{-1}(\nu_{K+L}(\overline{x})\big) + v\big(\nu_K^{-1}(\nu_{K+L}(\overline{x})\big) \ .$$

In view of the identity (8.1), the r.h.s. of the above equality equals $u \Box v(\overline{x})$, and (8.2) is proved.

Consider now the case $x^h \in \mathrm{int}(K + L) \ \forall h$. We set

$$y^h := \nabla w(x^h) = (\nabla(\varphi + \psi))^{-1}(x^h) \ ,$$

and we decompose x^h as $x^h_1 + x^h_2$, with

$$x^h_1 := \nabla \varphi(y^h) \in \mathrm{int}(K) \qquad \text{and} \qquad x^h_2 := \nabla \psi(y^h) \in \mathrm{int}(L) \ .$$

Then we have

$$u \Box v(x^h) = [\langle x^h_1, y^h \rangle - \varphi(y^h)] + [\langle x^h_2, y^h \rangle - \psi(y^h)] = u(x^h_1) + v(x^h_2) \ .$$

Let us now pass the the limit in h. By compactness, after possibly selecting a (not relabeled) subsequence, there exist $\lim_h x^h_1 =: \overline{x}_1 \in \partial K$ and $\lim_h x^h_2 =: \overline{x}_2 \in \partial L$. Since by assumption $u \in \mathcal{C}^0(K)$ and $v \in \mathcal{C}^0(L)$, we infer

$$\lim_h u \Box v(x^h) = u(\overline{x}_1) + v(\overline{x}_2)$$

In view of the identity (8.1), the above equality implies (8.2) provided

$$\overline{x}_1 = \nu_K^{-1}\big(\nu_{K+L}(\overline{x})\big) \qquad \text{and} \qquad \overline{x}_2 = \nu_L^{-1}\big(\nu_{K+L}(\overline{x})\big) \ .$$

In turn, by the $\mathcal{C}^2_+$ assumption on $\partial K, \partial L$, such conditions are satisfied provided the normal vectors $\nu_K(\overline{x}_1)$ and $\nu_L(\overline{x}_2)$ coincide. Let us show that in fact each of them agrees with

$$\overline{\xi} := \lim_h \frac{y^h}{\|y^h\|} \ .$$

Since $y^h = \nabla u(x_1^h)$, and $\|y^h\| \to +\infty$ (being $y^h = \nabla w_h(x^h)$ and $x^h \to \overline{x} \in \partial(K+L)$), by passing to the limit in the inequality

$$\frac{u(x)}{\|y^h\|} \geq \frac{u(x_1^h)}{\|y^h\|} + \langle \frac{y^h}{\|y^h\|}, x - x_1^h \rangle \ ,$$

we infer that any cluster point of the sequence $y^h/\|y^h\|$ belongs to the normal cone to ∂K at $\overline{x}_1$, which is reduced to $\nu_K(\overline{x}_1)$. In the same way we obtain $\overline{\xi} = \nu_L(\overline{x}_2)$, and the proof is achieved. □

Proof of Lemma 4.10. (i) Let $x \in \mathrm{dom}(u)$ be fixed. By the assumption $v(0) = 0$, we have $u_t(x) \leq u(x)$ for every $t > 0$, so that $\limsup_{t \to 0^+} u_t(x) \leq u(x)$. Let us prove that we also have

$$\liminf_{t \to 0^+} u_t(x) \geq u(x) \tag{8.3}$$

Assume $u, v \in \mathcal{L}'$, and set $\varphi := u^*$, $\psi := v^*$. We choose $r > \|\nabla u(x)\|$ and we set $c := \sup_{B_r} \psi$ (notice that c is finite because ψ is bounded on bounded sets [5, Theorem 4.4.13]). Then

$$\begin{aligned} u_t(x) &= \sup_{y \in \mathbb{R}^n} \big\{ \langle x, y \rangle - \varphi(y) - t\psi(y) \big\} \geq \sup_{y \in B_r} \big\{ \langle x, y \rangle - \varphi(y) \big\} - tc \\ &= \langle x, \nabla u(x) \rangle - \varphi(\nabla u(x)) - tc = u(x) - tc \,, \end{aligned}$$

and (8.3) follows by passing to the inferior limit as $t \to 0^+$.
Assume $u, v \in \mathcal{L}''$. Setting $L := \mathrm{dom}(v)$ and $m := \min v$, it holds $v \geq I_L + m$. Then

$$\begin{aligned} u_t(x) &= \inf_{x_1 + x_2 = x} \big\{ u(x_1) + t v(x_2/t) \big\} \geq \inf_{x_1 + x_2 = x} \big\{ u(x_1) + t I_L(x_2/t) \big\} + tm \\ &= \inf_{x_1 + x_2 = x} \big\{ u(x_1) + t I_{tL}(x_2) \big\} + tm = \inf_{x_1 \in K \cap (x - tL)} \{ u(x_1) \} + tm \,, \end{aligned}$$

and, thanks to the continuity of u at x, (8.3) follows by passing to the inferior limit as $t \to 0^+$.

Statement (ii) is an immediate consequence of the convexity of the functions u_t and of the differentiability of their pointwise limit u in the interior of its domain. □

Proof of Lemma 4.11. Set $K_t := \mathrm{dom}(u_t)$. First we claim that, for every fixed $x \in \mathrm{int}(K_t)$,

$$\text{the map } \ t \mapsto \nabla u_t(x) \ \text{ is differentiable on } (0, +\infty). \tag{8.4}$$

Indeed, as noticed in the proof of Lemma 4.9, the Fenchel conjugates $\varphi := u^*$ and $\psi := v^*$ are both of class $\mathcal{C}^2_+$ on $\mathbb{R}^n$. Therefore, the function $F : \mathbb{R}^n \times \mathbb{R}^n \times (0, +\infty) \to \mathbb{R}^n$ defined by

$$F(x, y, t) := \nabla \varphi(y) + t \nabla \psi(y) - x \ ,$$

is of class $\mathcal{C}^1$ on $\mathbb{R}^n \times \mathbb{R}^n \times (0, +\infty)$, and $\frac{\partial F}{\partial y} = \nabla^2 \varphi + t \nabla^2 \psi$ is nonsingular for every $y \in \mathbb{R}^n$. Consequently, by the implicit function theorem, the equation $F(x, y, t) = 0$ locally defines a map $y = y(x, t)$ which is of class $\mathcal{C}^1$ in its arguments. By Lemma 4.9, $(\mathrm{int}(K_t), u_t)$ is a convex function of Legendre type, hence by Proposition 2.2 ∇u_t is the inverse map of $\nabla \varphi_t$, namely

$$F(x, \nabla u_t(x), t) = \nabla \varphi_t(\nabla u_t(x)) - x = 0 \ .$$

Therefore, for every $x \in \mathrm{int}(K_t)$ and every $t > 0$, $y(x, t) = \nabla u_t(x)$, and (8.4) is proved.
Next, we apply again to Proposition 2.2 in order to write the identity

$$u_t(x) = \langle x, \nabla u_t(x) \rangle - \varphi_t\big(\nabla u_t(x)\big) \qquad \forall x \in \mathrm{int}(K_t) \ . \tag{8.5}$$

By (8.4) and (8.5) we obtain that, for every fixed $x \in \mathrm{int}(K_t)$, the map $t \mapsto u_t(x)$ is differentiable on $(0, +\infty)$, with

$$\frac{d}{dt} u_t(x) = \langle x, \frac{d}{dt}\big(\nabla u_t(x)\big)\rangle - \psi\big(\nabla u_t(x)\big) - \langle \nabla \varphi_t\big(\nabla u_t(x)\big), \frac{d}{dt}\big(\nabla u_t(x)\big)\rangle = -\psi\big(\nabla u_t(x)\big) \ .$$

$\square$

Proof of Lemma 4.12. We have

$$\begin{aligned} \int_{\mathbb{R}^n} \varphi(\nabla u(x))\, f(x)\, dx \ &= \int_{\mathbb{R}^n} \big(\langle x, \nabla u\rangle - u\big)\, f\, dx = -\int_{\mathbb{R}^n} \langle x, \nabla f\rangle\, dx + \int_{\mathbb{R}^n} f \log f\, dx \ . \\ &= -\int_{\mathbb{R}^n} \mathrm{div}(fx)\, dx + nJ(f) + \int_{\mathbb{R}^n} f \log f\, dx \ . \end{aligned}$$

We observe that

$$\int_{\mathbb{R}^n} \mathrm{div}(fx)\, dx = \lim_{r \to +\infty} \int_{B_r} \mathrm{div}(fx)\, dx = \lim_{r \to +\infty} r \int_{\partial B_r} f\, d\mathcal{H}^{n-1} = 0 \ ,$$

where the last equality holds true by Lemma 2.5. Therefore we have

$$\int_{\mathbb{R}^n} \varphi(\nabla u(x))\, f(x)\, dx = nJ(f) + \int_{\mathbb{R}^n} f \log f\, dx \ ,$$

and the lemma follows recalling that both $J(f)$ and $\int_{\mathbb{R}^n} f \log f\, dx$ are finite (*cf.* respectively Lemma 2.5 and Proposition 3.11). $\square$

Proof of Lemma 4.13. By definition we have

$$u_t(x) = \inf_{x_1 + x_2 = x} \Big\{ u(x_1) + tv\big(\frac{x_2}{t}\big) \Big\} \ .$$

Since

$$v_{\min} + I_L(x) \le v(x) \le v_{\max} + I_L(x) \qquad \forall x \in \mathbb{R}^n \ ,$$

it holds

$$\inf_{x_1 + x_2 = x} \Big\{ u(x_1) + tv_{\min} + tI_L\big(\frac{x_2}{t}\big) \Big\} \le u_t(x) \le \inf_{x_1 + x_2 = x} \Big\{ u(x_1) + tv_{\max} + tI_L\big(\frac{x_2}{t}\big) \Big\} \ ,$$

namely

$$tv_{\min} + \inf_{x_1 \in K \cap (x - tL)} \big\{ u(x_1) \big\} \le u_t(x) \le tv_{\max} + \inf_{x_1 \in K \cap (x - tL)} \big\{ u(x_1) \big\} \ .$$

Therefore the statement is satisfied by taking y as a point where u attains its minimum on $K \cap (x - tL)$. $\square$

Proof of Remark 4.7. By inspection of the proof of Theorems 4.5 and 4.6, one can see that assumption (4.1) is used only in Step 4 (in order to prove that $\Psi(0) < +\infty$) and in Step 5 (in order to prove that $\lim_{s \to 0^+} \Psi(s) = \Psi(0)$). Assume now $n = 1$, and drop assumption (4.1): let us indicate how Steps 4 and 5 (and consequently also Step 6) have to be modified in order to show that (4.3) continues to hold, possibly as an equality $+\infty = +\infty$.
In Step 4, we limit ourselves to prove that Ψ takes finite values at every $s > 0$.
In Step 5, the proof of the continuity of Ψ at every $s > 0$ remains unchanged, whereas for $s \to 0^+$ we make the following claim (whose proof is postponed below):

(8.6) if $\Psi(0) < +\infty$, then Ψ is continuous from the right at $s = 0$.

Consequently, in Step 6 we must distinguish two cases. In case $\Psi(0) < +\infty$, thanks to (8.6) equality (4.4) can be proved exactly as before. In case $\Psi(0) = +\infty$, (4.4) continues to hold as an equality

$+\infty = +\infty$, and it can be proved by slight modifications of the case $\Psi(0) < \infty$. More precisely, (4.22) and (4.24) in Step 6 remain unchanged, whereas (4.23) has to be replaced by

$$\liminf_{s\to 0^+} \Psi(s) \geq \sup_{C\subset\subset E} \liminf_{s\to 0^+} \int_C \psi \, d\mu(f_s) = \sup_{C\subset\subset E} \int_C \psi d\mu(f) = +\infty \tag{8.7}$$

(notice that the second equality in (8.7) holds by dominated convergence, since by Lemma 4.10 we have $\psi(\nabla u_s) f_s \to \psi(\nabla u) f$ as $s \to 0^+$, and by Lemma 3.9 and Lemma 4.10 (ii) the nonnegative functions $\psi(\nabla u_s)\, f_s$ are bounded above on C by some continuous function independent of s).
Let us finally prove (8.6). Assume

$$\Psi(0) = \int_{\mathbb{R}} \psi\varphi'' e^{\varphi - y\varphi'} \, dy < +\infty \ . \tag{8.8}$$

Since $n = 1$, (4.13) simplifies into

$$\Psi(s) = \Psi_0(s) + s\Psi_1(s) \ ,$$

where

$$\begin{aligned} \Psi_0(s) &:= \int_{\mathbb{R}} h_0(s,y)\, dy \qquad & h_0(s,y) &:= \psi e^{\varphi_s - y\varphi_s'} \varphi'' \chi_{Q_s} \\ \Psi_1(s) &:= \int_{\mathbb{R}} h_1(s,y)\, dy \qquad & h_1(s,y) &:= \psi e^{\varphi_s - y\varphi_s'} \psi'' \chi_{Q_s} \ . \end{aligned}$$

To get (8.6) it suffices to show that

$$\lim\nolimits_{s\to 0^+} \Psi_0(s) = \Psi(0) \tag{8.9}$$

$$\lim\nolimits_{s\to 0^+} s\Psi_1(s) = 0 \ . \tag{8.10}$$

Thanks to assumption (8.8), (8.9) can be proved exactly as before (*cf.* the proof of (4.16)). To prove (8.10), we write

$$s\Psi_1(s) = I_+(s) + I_-(s) := \int_{\mathbb{R}_+} sh_1(s,y)\, dy + \int_{\mathbb{R}_-} sh_1(s,y)\, dy \ ,$$

and we show that both $I_\pm(s)$ are infinitesimal as $s \to 0^+$. Let us consider $I_+(s)$ (the case of $I_-(s)$ is completely analogous).
We observe that

$$0 \leq sh_1(s,y) = s\psi e^{\varphi_s - y\varphi_s'} \psi'' \chi_{Q_s} \leq -F(y) G_s'(y) \qquad \forall y, s > 0 \ ,$$

where we have set

$$F(y) := \frac{\psi}{y} e^{\varphi - y\varphi'} \qquad \text{and} \qquad G_s(y) := e^{s(\psi - y\psi')} \ .$$

Then an integration by parts gives

$$0 \leq I_+(s) \leq \lim_{\varepsilon\to 0^+,\ r\to+\infty} \Big\{ \int_\varepsilon^r F'(y) G_s(y)\, dy + F(\varepsilon) G_s(\varepsilon) - F(r) G_s(r) \Big\} \ .$$

Since $\psi(0) = \psi'(0) = 0$ (respectively because $v \geq 0$ and $\psi \geq 0$), passing to the limit in ε gives

$$0 \leq I_+(s) \leq \lim_{r\to+\infty} \Big\{ \int_0^r F'(y) G_s(y)\, dy - F(r) G_s(r) \Big\} \ . \tag{8.11}$$

Next we observe that the following limit exists:

$$\alpha := \lim_{r\to+\infty} F(r) = \lim_{r\to+\infty} \int_0^r F'(y) dy \ .$$

Indeed a straightforward computation gives

$$F'(y) = -\psi\varphi'' e^{\varphi - y\varphi'} + \frac{e^{\varphi - y\varphi'}}{y^2}(\psi' y - \psi) \ ,$$

and both the functions at the right and side are integrable on $(0, +\infty)$ (the former by assumption (8.8), the latter because it is nonnegative).

Let us show that $\alpha > 0$ cannot occur. Indeed in such case, for some constants $\overline{c}$ and $\overline{r}$, it would be $F(r) \geq \overline{c}\ \forall r \geq \overline{r}$. This would contradict (8.8), since

$$\Psi(0) \geq \int_{\overline{r}}^{+\infty} \psi\varphi'' e^{\varphi - y\varphi'}\, dy \geq \overline{c} \int_{\overline{r}}^{+\infty} y\varphi''\, dy = \overline{c}\left\{ \lim_{r\to+\infty} [r\varphi'(r) - \varphi(r)] - [\overline{r}\varphi'(\overline{r}) - \varphi(\overline{r})] \right\} = +\infty \ .$$

Taking into account that $\alpha = 0$ (and also that $\lim_{r\to+\infty} G_s(r) = 0$), we may rewrite (8.11) as

$$0 \leq I_+(s) \leq \lim_{r\to+\infty} \int_0^r F'(y) G_s(y)\, dy \ . \tag{8.12}$$

Moreover, since $\alpha = 0$, we have in particular $\int_0^{+\infty} F'(y)\, dy < +\infty$, which implies $F' \in L^1(0, +\infty)$. Therefore, for every fixed $s > 0$, the functions $F' G_s$ satisfy

$$|F'(y) G_s(y)| \leq |F'(y)| \in L^1(0, +\infty) \ .$$

We deduce that (8.12) can be rewritten as

$$0 \leq I_+(s) \leq \int_0^{+\infty} F'(y) G_s(y)\, dy \ . \tag{8.13}$$

Finally, passing to the limit as $s \to 0^+$ in the right hand side of (8.13) we obtain

$$\lim_{s\to 0^+} \int_0^{+\infty} F'(y) G_s(y)\, dy = \int_0^{+\infty} \lim_{s\to 0^+} F'(y) G_s(y)\, dy = \int_0^{+\infty} F'(y)\, dy = \alpha = 0 \ .$$

This implies that $I_+(s)$ is infinitesimal as $s \to 0^+$ and the proof is achieved. □

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